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Performance analysis of a GI/M/1 queue with single working vacation

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ABSTRACT

Consider a GI/M/1 queue with single working vacation. During the vacation period, the server works at a lower rate rather than stopping completely, and only takes one vacation each time. Using the matrix analytic approach, the steady-state distributions of the number of customers in the system at both arrival and arbitrary epochs are obtained. Then the closed property of the conditional probability of gamma distribution is proved and using it the waiting time of an arbitrary customer is analyzed. Finally, Some numerical results and effect of critical model parameters on performance measures have been presented.

1. Introduction

Working vacation is one kind of policy under which the server can provide the service at a lower speed during the vacation period rather than stopping completely, and was introduced by Servi and Finn [13] in 2002, where the authors established an M/M/1 queue to model an Optical network. This working vacation system belongs to the scope of the vacation queue research, but is different from the classical vacation system in which the server completely stops working during the vacation period. The classical vacation queues have been well investigated and extensively used in modelling the computer networks, communication systems and production management et al. Details can be seen in the surveys of Doshi [5,6] and the monographs of Takagi [14] and Tian and Zhang [15].

As stated above, based on the classical vacation queues, Servi and Finn [13] first studied an M/M/1 queue with the working vacation policy and illustrated the analysis of a WDM optical access network using multiple wavelengths which can be reconfigured. In fact, this type of models can be shown as systems with lower and higher speeds, and is attracting growing interest because they can represent many practical problems. Some literatures on various types of working vacation queues have emerged.

- On the M/G/1-type systems with the Poisson arrival, Kim et al. [7], Wu and Takagi [18] gave the analysis of different special models by the decomposition method and Laplace Stieltjes transform (LST) method, respectively. And, Yi et al. [19] considered a discrete-time Geo/G/1 working vacation system using the results of the corresponding queue with disasters. Recently, Li et al. [10,11] applied the matrix analytic method to analyze this type of working vacation queues.
- On the GI/M/1-type systems with the general arrival, Baba [1] first presented a GI/M/1 queue with multiple working vacations using the matrix analytic method. Li et al. [8,9] considered two discrete-time GI/Geo/1 working vacation queues. And, a finite-buffer GI/M/1/N queue with multiple working vacations can be seen in the survey of Banik et al. [2].

From the literatures listed above, we discovered that all research efforts focus on the multiple working vacation (MWV) policy, in other words, the server may take vacations more than one time when he enters into vacation each time. Thus, we

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will give the analysis of the GI/M/1 queue with the single working vacation. For the GI/M/1 queues with classical vacations, Chatterjee and Mukherjee [4] considered a multiple vacation model, and Tian et al. [16] and Tian and Zhang [15,17] presented the analysis of a series of GI/M/1 models by the matrix analytic method. Recently, Chae et al. [3] obtained the stochastic decomposition structures for the GI/M/1 queues with single vacation.

In this paper, we provide a detailed discussion of a GI/M/1 queue with single working vacation (SWV), denoted by GI/M/1/SWV in which the server enters into vacation when there are no customers and takes service at a lower rate during the vacation period. Meanwhile, he only takes one vacation each time, and must come back to the normal working level no matter whether there are customers at the vacation ending instant. If there are customers when the vacation ends, the server begins to serve one customer at the normal rate immediately; otherwise, he will stay in an idle period. Thus, the idle period is the duration of time when the server is available for service but there are no customers present. The server may stay in the busy period, vacation period and idle period and the model is more complex than that with multiple working vacations analyzed in an earlier paper by Baba [1].

Such a working vacation policy has many significant applications. For example, under management structures, single vacation each time is more reasonable for employees, and in factories, if the repair duration of the machines can be seen as the vacation period, the multiple working vacation policy is not appropriate to model this system. Meanwhile, under the single working vacation policy, this system is the model with alternations of higher and slower speed periods, and is the more appropriate simulation to the practical cases. When there are fewer jobs (units), a slower speed period begins to economize the system.

We will analyze the transition situations and obtain the transition probability matrix for an embedded Markov chain. Using the matrix-geometric approach, we give concise expressions for distributions of the queue length at the arrival epoch and arbitrary epoch. Of same important is to verify and first apply the closed property of conditional probability of Γ-distributions to analyze the waiting time distribution of the GI/M/1/SWV. Although such a property can be proved easily, its application to the waiting time analysis has certain theoretical implication. The stochastic decomposition structures for the queue length and waiting time are also presented, but such property is not used in Baba[1].

The rest of this paper is organized as follows. Section 2 provides the model formulation and the transition analysis for an embedded Markov chain. Section 3 is devoted to obtain the steady-state queue length distributions at arrival and arbitrary epochs. Section 4 provides the technical foundation, that is, the closed property of conditional probability of Γ-distributions, and results for the waiting time. Section 5 gives some numerical results and Section 6 concludes the paper.

2. Model formulation and embedded markov chain

Consider a GI/M/1 queue such that its arrival process is a general distribution process. The model is denoted in detail below:

1. The inter-arrival times $\{T_n, n \geqslant 1\}$ are independent and identically distributed (IID) with a general distribution function, denoted by $A(t)$ with a mean $1/\lambda$ and a Laplace Stieltjes transform (LST), denoted by $a^*(s)$.
2. The server begins a vacation each time when the queue becomes empty and if there are customers arriving during a vacation period, the server continues to work at a lower rate. The working vacation period is an operation period at a lower speed. These assumptions are the same with those in Baba [1].
3. The server only can take one vacation each time and will come back to the normal working level no matter whether there are customers. Thus, it is possible that the server stays in an idle period. As stated above, the idle period is the duration of time when the server is available for service but there are no customers present. The server may stay in the busy period, vacation period and idle period.
4. The service times during a service period, the service times during a working vacation and the working vacation times are exponentially distributed with rate μ, η and θ, respectively. The service time, arrival process and vacation times are mutually independent. The customers are served on a first-come, first-served basis by the server.

Suppose τ_n the arrival epoch of nth customers with $\tau_0 = 0$. Let $L(t)$ be the number of customers in the system at time t and $L_n = L(\tau_n - 0)$ be the number of the customers before the nth arrival. Define

$$J_n = \begin{cases} 1, & \text{the } n\text{-th arrival occurs during a service or idle period}, \\ 0, & \text{the } n\text{-th arrival occurs during a working vacation period}. \end{cases}$$

The process $\{(L_n, J_n), n \geqslant 1\}$ is an embedded Markov chain with the state space

$$\Omega = \{(k,j), k \geqslant 0, j = 0, 1\},$$

where $(0,1)$ represents that the server stays in the idle period when a new customer arrives.

In order to express the transition matrix of (L_n, J_n), we introduce the probability measure:

$$\mathrm{P}_{(i,j),(k,l)} = \mathrm{P}(L_{n+1} = k, J_{n+1} = l | L_n = i, J_n = j).$$

Now, we consider the transition probabilities of (L_n, J_n). First, considering the case during a service period, the transition from $(i,1)$ to $(j,1)$ occurs if $i+1-j$ services are completed during an inter-arrival time. Therefore, we have

$$\mathrm{P}_{(i,1),(j,1)} = \int_0^{\infty} e^{-\mu t}\frac{(\mu t)^{i+1-j}}{(i+1-j)!}dA(t) = b_{i+1-j}, \quad 1 \leqslant j \leqslant i+1. \tag{1}$$

Second, the transition from $(i,1)$ to $(0,1)$ occurs only if $i+1$ customers are served, the server enters into the vacation period and the vacation has ended before the next arrival. Thus,

$$\begin{aligned}\mathrm{P}_{(i,1),(0,1)} &= \mathrm{P}\left\{A \geqslant \sum_{k=0}^{i+1} S_k + V\right\} = \int_0^{\infty}\int_0^{t} \theta e^{-\mu x}\left[1-\sum_{k=0}^{i}\frac{(\mu(t-x))^k}{k!}e^{-\mu(t-x)}\right]dxdA(t) \\ &= 1-a^*(\theta)-\sum_{k=0}^{i}\int_0^{+\infty}\int_0^{t}\theta e^{-\theta x}\frac{(\mu(t-x))^k}{k!}e^{-\mu(t-x)}dxdA(t) = 1-a^*(\theta)-\sum_{k=0}^{i}c_k, \quad 1\leqslant j\leqslant i+1,\end{aligned} \tag{2}$$

where it is assumed that S_k is the service time of the kth customer with the normal rate μ and $S_0 \equiv 0$, V is the working vacation time and A stands for the limit of T_n as $n \to \infty$.

The transition from $(i,1)$ to *(0,0)* occurs only if $i+1$ customers are served, the server enters into the vacation period and the vacation has not ended before the next arrival. Thus,

$$\begin{aligned}\mathrm{P}_{(i,1),(0,0)} &= \mathrm{P}\left\{\sum_{k=0}^{i+1} S_k < A < \sum_{k=0}^{i+1} S_k + V\right\} = \int_0^{\infty}\int_0^{+\infty}\theta e^{-\theta x}\,\mathrm{P}\left\{t-x<\sum_{k=0}^{i+1}S_k<t\right\}dxdA(t) \\ &= \int_0^{\infty}\int_0^{t}\theta e^{-\theta x}\left[\sum_{k=0}^{i}\frac{(\mu(t-x))^k}{k!}e^{-\mu(t-x)}-\sum_{k=0}^{i}\frac{(\mu t)^k}{k!}e^{-\mu t}\right]dxdA(t)+\int_0^{\infty}e^{-\theta t}\left[1-\sum_{k=0}^{i}\frac{(\mu t)^k}{k!}e^{-\mu t}\right]dA(t) \\ &= a^*(\theta)+\sum_{k=0}^{i}c_k-\sum_{k=0}^{i}b_k, \qquad j=i+1.\end{aligned} \tag{3}$$

Consider the transition from $(i,0)$ to $(j,0)$. It arises if the next customer arrives during the vacation period and $i+1-j$ customers are served during one inter-arrival time. Then,

$$\mathrm{P}_{(i,0),(j,0)} = \int_0^{\infty} e^{-\theta t}\frac{(\eta t)^{i+1-j}}{(i+1-j)!}e^{-\eta t}dA(t) = d^{(1)}_{i+1-j}, \qquad i\geqslant 0, 1\leqslant j\leqslant i+1. \tag{4}$$

Similarly, the transition from $(i,0)$ to $(j,1)$ occurs only if the vacation ends and there are $i+1-j$ customers served during one inter-arrival time. Thus, for $i \geqslant 0, 1 \leqslant j \leqslant i+1$

$$\mathrm{P}_{(i,0),(j,1)} = \int_0^{\infty}\int_0^{t}\theta e^{-\theta x}\sum_{k=0}^{i+1-j}\frac{(\eta x)^k}{k!}e^{-\eta x}\frac{(\mu(t-x))^{i+1-j-k}}{(i+1-j-k)!}e^{-\mu(t-x)}dxdA(t) = v_{i+1-j}. \tag{5}$$

There are two possible cases to cause the transition from $(i,0)$ to $(0,0)$. Case 1: if the residual vacation time is longer than one inter-arrival time, more than $i+1$ customers can be served during the inter-arrival time. Case 2: if the residual vacation time is not longer than one inter-arrival time, all $i+1$ customers are served during the inter-arrival time and the next arrival occurs during another vacation period. Therefore,

$$\begin{aligned}\mathrm{P}_{(i,0),(0,0)} &= \int_0^{\infty}e^{-\theta t}\left[1-\int_t^{\infty}\frac{\eta(\eta u)^i}{i!}e^{-\eta u}du\right]dA(t)+\sum_{k=0}^{i}\int_0^{\infty}\int_0^{t}\theta e^{-\theta x}\frac{(\eta x)^k}{k!}e^{-\eta x}\int_0^{t-x}\frac{\mu(\mu u)^{i-k}}{(i-k)!}e^{-\mu u}e^{-\theta(t-x-u)}dudxdA(t) \\ &= a^*(\theta)-\sum_{k=0}^{i}d_k^{(1)}+m_i,\end{aligned}$$

where we denote

$$m_i = \sum_{k=0}^{i}\int_0^{\infty}\int_0^{t}\theta e^{-\theta x}\frac{(\eta x)^k}{k!}e^{-\eta x}\int_0^{t-x}\frac{\mu(\mu u)^{i-k}}{(i-k)!}e^{-\mu u}e^{-\theta(t-x-u)}dudxdA(t), \quad i\geqslant 0,$$

and, the equality

$$\int_t^{\infty}\frac{\eta(\eta u)^i}{i!}e^{-\eta u}dudA(t) = \sum_{k=0}^{i}d_k^{(1)}$$

is obvious after some transposition.

Similarly, the transition from $(i,0)$ to $(0,1)$ occurs under two cases: Case 1: all $i+1$ customers are served completely during the vacation time and the next arrival occurs in the idle period. Case 2: there are $k(k < i+1)$ customers served during the vacation time and the residual customers are served during the service period, and the server enters into vacation, then the next arrival occurs during the idle period. Then, we have

$$\begin{aligned}
\mathrm{P}_{(i,0),(0,1)} &= \int_0^\infty \int_0^t \theta e^{-\theta x} \int_0^{t-x} \frac{\eta(\eta u)^i}{i!} e^{-\eta u} du dx dA(t) + \int_0^\infty \int_0^t \theta e^{-\theta x} \sum_{k=0}^{i} \frac{(\eta x)^k}{k!} e^{-\eta x} \int_0^{t-x} \frac{\mu(\mu u)^{i-k}}{(i-k)!} \\
&\quad \times e^{-\mu u}(1-e^{-\theta(t-x-u)}) du dx dA(t) \\
&= 1 - a^*(\theta) - m_i \\
&\quad - \int_0^\infty \int_0^t \theta e^{-\theta x} \sum_{k=0}^{i} \frac{(\eta x)^k}{k!} e^{-\eta x} \sum_{j=0}^{i-k} \frac{(\mu(t-x))^j}{j!} e^{-\mu(t-x)} du dx dA(t) = 1 - a^*(\theta) - m_i - \sum_{k=0}^{i} v_k, \quad i \geqslant 0.
\end{aligned}$$

In the above equation, we use the equality:

$$\sum_{k=0}^{i} v_k = \int_0^\infty \int_0^t \theta e^{-\theta x} \sum_{k=0}^{i} \frac{(\eta x)^k}{k!} e^{-\eta x} \sum_{l=0}^{i-k} \frac{(\mu(t-x))^l}{l!} e^{-\mu(t-x)} dx dA(t).$$

Using the lexicographical sequence for the states, the transition probability matrix of (L_n,J_n) can be written as the Block–Jacobi matrix

$$\widetilde{\mathrm{P}} = \begin{bmatrix} B_0 & A_0 & & & & \\ B_1 & A_1 & A_0 & & & \\ B_2 & A_2 & A_1 & A_0 & & \\ B_3 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{6}$$

where

$$A_k = \begin{bmatrix} d_k^{(1)} & v_k \\ 0 & b_k \end{bmatrix}, \quad k \geqslant 0;$$

$$B_k = \begin{bmatrix} a^*(\theta) - \sum_{i=0}^{k} d_i^{(1)} + m_k & 1 - a^*(\theta) - m_k - \sum_{i=0}^{k} v_i \\ a^*(\theta) + \sum_{i=0}^{k} c_i - \sum_{i=0}^{k} b_i & 1 - a^*(\theta) - \sum_{i=0}^{k} c_i. \end{bmatrix}, \quad k \geqslant 0.$$

Evidently, $\widetilde{\mathbf{P}}$ is a stochastic matrix and its structure indicates that (L_n,J_n) is irreducible and aperiodic.

3. Steady-state distributions

We will derive the steady-state distribution of (L_n,J_n) at the arrival epoch by using Neut's matrix-geometric approach. In order to derive the steady-state distribution, we need the following lemma.

Lemma 1. *If $\rho = \lambda/\mu < 1$ and $\theta > 0$, the matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$ has the minimal nonnegative solution*

$$R = \begin{bmatrix} \gamma & \alpha(\xi-\gamma) \\ 0 & \xi \end{bmatrix},$$

where ξ and γ are the unique roots in the range $0 < z < 1$ of the equations $z = a^*(\mu(1-z))$ and $z = a^*(\theta + \eta(1-z))$, respectively and

$$\alpha = \frac{\theta}{\theta - (\mu-\eta)(1-\gamma)}.$$

From the fact that all matrices A_k are upper triangular, R can be assumed with the same structure and substituting R into the matrix equation, we can obtain the minimal nonnegative solution R. The detailed proof is similar to that of lemma 2 in Baba [1] and we omit it here.

Theorem 1. *The Markov chain $\widetilde{\mathrm{P}}$ is positive recurrent if and only if $\rho < 1$ and $\theta > 0$.*

Proof. Based on Theorem 1.5.1 of Neuts [12] and Baba [1], the Markov chain $\widetilde{\mathrm{P}}$ is positive recurrent if and only if the spectral radius SP $(R) = \max\{\gamma,\xi\}$ of the rate matrix R is less than 1, and the matrix $B[R] = \sum_{k=0}^{\infty} R^k B_k$has a positive left invariant vector. Evidently, SP $(R) = \max\{\gamma,\xi\} < 1$. Substituting R and B_k into $B[R]$, we obtain

$$B[R] = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

where

$$H_{11} = \sum_{k=0}^{\infty}\left[\gamma^k\Psi_k + \alpha(\xi-\gamma)\sum_{j=0}^{k-1}\xi^j\gamma^{k-1-j}\left(a^*(\theta)+\sum_{i=0}^{k}(c_i-b_i)\right)\right],$$

$$H_{12} = \sum_{k=0}^{\infty}\left[\gamma^k\Phi_k + \alpha(\xi-\gamma)\sum_{j=0}^{k-1}\xi^j\gamma^{k-1-j}\left(1-a^*(\theta)-\sum_{i=0}^{k}c_i\right)\right],$$

$$H_{21} = \sum_{k=0}^{\infty}\xi^k\left[a^*(\theta)+\sum_{i=0}^{k}(c_i-b_i)\right],\quad H_{22} = \sum_{k=0}^{\infty}\xi^k\left[1-a^*(\theta)-\sum_{i=0}^{k}c_i\right].$$

Assume

$$\Psi_k = a^*(\theta) - \sum_{i=0}^{k} d_i^{(1)} + m_k;\quad \Phi_k = 1-a^*(\theta)-\sum_{i=0}^{k} v_i - m_k.$$

We have

$$\sum_{k=0}^{\infty}\sum_{i=0}^{k} c_i\xi^k = \sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\xi^k\int_0^{\infty}\int_0^t \theta e^{-\theta x}\frac{(\mu(t-x))^i}{i!}e^{-\mu(t-x)}dxdA(t) = \frac{1}{1-\xi}\int_0^{\infty}\int_0^t \theta e^{-\theta x}e^{-\mu(t-x)(1-\xi)}dxdA(t)$$
$$= \frac{1}{1-\xi}\frac{\theta(\xi-a^*(\theta))}{\theta-\mu(1-\xi)}.$$

It is easy to obtain the expressions for H_{21} and H_{22}

$$H_{22} = \frac{\theta-\mu(1-a^*(\theta))}{\theta-\mu(1-\xi)} = \sigma;\qquad H_{21} = 1-\sigma.$$

Meanwhile, to compute the expressions for H_{11}, H_{12}, we first have

$$\sum_{k=0}^{\infty}\sum_{i=0}^{k} v_i\gamma^k = \frac{1}{1-\gamma}\sum_{i=0}^{\infty} v_i\gamma^i = \frac{\alpha(a^*(\mu(1-\gamma))-\gamma)}{1-\gamma},$$

$$\sum_{k=0}^{\infty}\gamma^k\sum_{i=0}^{k} d_i^{(1)} = \frac{1}{1-\gamma}\int_0^{\infty} e^{-(\theta+\eta(1-\gamma))t}dA(t) = \frac{\gamma}{1-\gamma},$$

$$\sum_{k=0}^{\infty} m_k\gamma^k = \sum_{i=0}^{\infty}\int_0^{\infty}\int_0^t \theta e^{-\theta x}\frac{(\eta\gamma x))^i}{i!}e^{-\eta x}\int_0^{t-x}\sum_{k=i}^{\infty}\frac{\mu(\gamma\mu u)^{k-i}}{(k-i)!}e^{-\mu u}e^{-\theta(t-x-u)}dudxdA(t)$$
$$= \int_0^{\infty}\int_0^t \theta e^{-(\theta+\eta(1-\gamma))x}e^{-\theta(t-x)}\int_0^{t-x}\mu e^{(\theta-\mu(1-\gamma))u}dudxdA(t)$$
$$= \frac{\mu}{\theta-\mu(1-gamma)}\left[\alpha(a^*(\mu(1-\gamma))-\gamma)-\frac{\theta}{\eta(1-\gamma)}(a^*(\theta)-\gamma)\right].$$

And,

$$\alpha(\xi-\gamma)\sum_{k=0}^{\infty}\sum_{j=0}^{k-1}\xi^j\gamma^{k-1-j}\sum_{i=0}^{k}c_i = \alpha\sum_{i=0}^{\infty}c_i\sum_{k=0}^{i}(\xi^k-\gamma^k) = \alpha\left[\frac{1}{1-\xi}\frac{\theta(\xi-a^*(\theta))}{\theta-\mu(1-\xi)}-\frac{1}{1-\gamma}\frac{\theta(a^*(\mu(1-\gamma))-a^*(\theta))}{\theta-\mu(1-\gamma)}\right].$$

Therefore,

$$H_{11} = (1-\alpha\frac{\mu}{\eta})\frac{a^*(\theta)-\gamma}{1-\gamma}-\alpha\frac{\mu(a^*(\theta)-\xi)}{\theta-\mu(1-\xi)} = \delta,\quad H_{12} = 1-\delta.$$

It can be easily verified that $B[R]$ has the left invariant vector

$$K^*(1-\sigma, 1-\delta), \tag{7}$$

where K^* is any positive integer. Thus, if $\rho < 1$, the Markov chain $\widetilde{P}$ is positive recurrent. □

If $\rho < 1$, let (L,J) be the stationary limit of the process (L_n,J_n). Denote

$$\pi_k = (\pi_{k0}, \pi_{k1}),\quad k \geqslant 0;\quad \pi_{kj} = \lim_{n\to\infty} \mathrm{P}\{L_n = k, J_n = j\},\quad (k,j)\in\Omega.$$

Theorem 2. *If $\rho < 1$, the stationary probability distribution of (L,J) is*

$$\begin{cases} \pi_{k0} = K(1-\sigma)\gamma^k, & k \geqslant 0, \\ \pi_{k1} = K\Big[(1-\delta)\xi^k + (1-\sigma)\alpha(\xi^k - \gamma^k)\Big], & k \geqslant 0, \end{cases} \tag{8}$$

where

$$K = \frac{(1-\gamma)(1-\xi)}{(1-\delta)(1-\gamma) + (1-\sigma)(1-\xi+\alpha(\xi-\gamma))} = \beta(1-\gamma)(1-\xi).$$

Proof. Using Theorem 1.5.1 of Neuts [12], (π_{00}, π_{01}) is given by the positive left invariant vector (7), thus we have

$$\pi_k = (\pi_{k0}, \pi_{k1}) = (\pi_{00}, \pi_{01})R^k, \quad k \geqslant 0. \tag{9}$$

Taking (π_{00}, π_{01}) and R^k into (9), we easily obtain the expressions for π_{k0} and π_{k1}. K can be derived using the normalizing condition $\sum_{k=0}^{\infty}(\pi_{k0} + \pi_{k1}) = 1$. □

The state probabilities of a server in the steady-state are shown as:

$$\begin{aligned} &\mathrm{P}\{J=0\} = \sum_{k=0}^{\infty} \pi_{k0} = \beta(1-\sigma)(1-\xi), \\ &\mathrm{P}\{\text{The server is in the idle period}\} = \pi_{01} = \beta(1-\delta)(1-\gamma)(1-\xi), \\ &\mathrm{P}\{\text{The server is in the busy period}\} = \sum_{k=1}^{\infty} \pi_{k1} = \beta[(1-\sigma)\alpha(\xi-\gamma) + (1-\delta)\xi(1-\gamma)]. \end{aligned} \tag{10}$$

From Theorem 2, the distribution of the stationary queue length L is

$$\mathrm{P}\{L=k\} = \pi_{k0} + \pi_{k1} = K\Big[(1-\sigma)\gamma^k + (1-\delta)\xi^k + (1-\sigma)\alpha(\xi^k - \gamma^k)\Big], \qquad k \geqslant 0.$$

Theorem 3. *If $\rho < 1$ and $\mu > \eta$, the stationary queue length L can be decomposed into the sum of two independent random variables: $L = L_0 + L_d$, where L_0 is the stationary queue length of a classical GI/M/1 queue without vacation, and follows a geometric distribution with parameter $1-\xi$, and the additional queue length $L_d$has a modified geometric distribution*

$$\begin{aligned} &\mathrm{P}\{L_d = 0\} = \beta(1-\gamma)(1-\delta+1-\sigma) = q \\ &\mathrm{P}\{L_d = k\} = (1-q)(1-\gamma)\gamma^{k-1}, \qquad k \geqslant 1. \end{aligned} \tag{11}$$

Proof. From the probability expression for L above, the probability generating function (PGF) of L is shown as follows

$$\begin{aligned} Q(z) &= \sum_{k=0}^{\infty} z^k \pi_{k0} + \sum_{k=0}^{\infty} z^k \pi_{k1} = K\left[\frac{1-\sigma}{1-\gamma z} + \frac{1-\delta}{1-\xi z} + (1-\sigma)\alpha(\xi-\gamma)\frac{1}{1-\xi z}\frac{z}{1-\gamma z}\right] \\ &= \frac{1-\xi}{1-\xi z}\beta(1-\gamma)\left[1-\delta + \frac{(1-\sigma)}{1-\gamma z}(1-\xi z) + (1-\sigma)\alpha(\xi-\gamma)\frac{z}{1-\gamma z}\right] \\ &= \frac{1-\xi}{1-\xi z}\beta(1-\gamma)\left[1-\delta+1-\sigma + (1-\sigma)(\alpha-1)(\xi-\gamma)\frac{z}{1-\gamma z}\right] = \frac{1-\xi}{1-\xi z}\left[q + (1-q)\frac{(1-\gamma)z}{1-\gamma z}\right] \\ &= L_0(z)L_d(z). \qquad \square \end{aligned} \tag{12}$$

Eq. (11) indicates that the additional delay L_d can be written as the mixture of two random variables: $L_d = q\,X_0 + (1-q)X_1$, where $X_0 \equiv 0$, and X_1 follows a geometric distribution with parameter $(1-\gamma)$.

The mean stationary queue length is below

$$E(L) = \frac{\xi}{1-\xi} + \beta(1-\sigma)(\alpha-1)(\xi-\gamma)\frac{1}{1-\gamma} = \frac{\xi}{1-\xi} + \frac{\mu-\eta}{\theta}\frac{(1-\sigma)\alpha(\xi-\gamma)}{(1-\delta)\xi(1-\gamma)+(1-\sigma)\alpha(\xi-\gamma)}.$$

We also can obtain the queue length distribution at the arbitrary epochs. We define that $p_k = \lim_{t\to\infty}\mathrm{P}\,\{L(t) = k\}, \quad k \geqslant 0$.

Theorem 4. *If $\rho < 1$ and $\theta > 0$, the limiting distribution of $L(t)$ exists. And, we obtain*

$$\begin{cases} p_0 = 1 - \beta\frac{\lambda}{\mu}\{(1-\delta+(1-\sigma)\alpha)(1-\gamma) - \frac{\mu-\eta}{\theta}(1-\sigma)\alpha[1-a^*(\mu(1-\gamma))](1-\xi)\}, \\ p_k = \beta(1-\xi)(1-\gamma)\frac{\lambda}{\mu}\Big\{(1-\delta+(1-\sigma)\alpha)\xi^{k-1} - \frac{\mu-\eta}{\theta}(1-\sigma)\alpha[1-a^*(\mu(1-\gamma))]\gamma^{k-1}\Big\}, \\ k \geqslant 1. \end{cases}$$

Using the method of the semi-Markov process (SMP), we can obtain the result directly. The proof is similar to that in Baba [1], and we omit it here.

Remark 1. If $\eta = 0$, i.e., the server can not take service in the vacation period, we can obtain the results in the GI/M/1 queue with classical single vacation.

4. Waiting time analysis

4.1. Conditional probability for Γ- distributions

To analyze the steady-state waiting time, we first demonstrate the closed property of conditional probability for Γ-distributions.

Assume that V follows an exponential distribution with parameter θ, and X follows a Γ-distribution with parameters α and η, i.e., its probability density function is

$$f_X(t) = \frac{\eta^\alpha t^{\alpha-1}}{\Gamma(\alpha)} e^{-\eta t}, \quad t > 0, \quad \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

We have two lemmas to present the closed property of conditional probability for Γ-distributions.

Lemma 4.1. *If X and V are independent mutually, under the condition $X < V$, X follows the Γ-distribution with parameters α and $\theta + \eta$.*

Proof. First, we compute the conditional probability

$$\mathrm{P}\{X < V\} = \int_0^\infty \mathrm{P}\{V > t\} \frac{\eta^\alpha t^{\alpha-1}}{\gamma(\alpha)} e^{-\eta t} dt = \frac{\eta^\alpha}{\gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(\eta+\theta)t} dt = \frac{\eta^\alpha}{\gamma(\alpha)(\theta+\eta)^\alpha} \int_0^\infty u^{\alpha-1} e^{-ut} du = \left(\frac{\eta}{\theta+\eta}\right)^\alpha;$$

$$\mathrm{P}\{X < x, X < V\} = \frac{\eta^\alpha}{\gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-(\theta+\eta)t} dt.$$

Then,

$$\mathrm{P}\{X < x | X < V\} = \frac{\mathrm{P}\{X < x, X < V\}}{\mathrm{P}\{X < V\}} = \int_0^x \frac{(\theta+\eta)^\alpha}{\gamma(\alpha)} t^{\alpha-1} e^{-(\theta+\eta)t} dt.$$

Thus, under the condition $X < V$, X follows Γ-distribution with parameters α and $\theta + \eta$. □

Meanwhile, assuming Y follow an exponential distribution with parameter η, we can verify the closed property of another conditional probability.

Lemma 4.2. *If X and V are independent mutually, under the condition $X \leqslant V < X + Y$, V follows Γ-distribution with parameters $\alpha + 1$ and $\theta + \eta$.*

Proof. Similarly, we have

$$\mathrm{P}\{X \leqslant V < X + Y\} = \int_0^\infty \mathrm{P}\{t \leqslant V < t + Y\} \frac{\eta^\alpha t^{\alpha-1}}{\gamma(\alpha)} e^{-\eta t} dt = \frac{\eta^\alpha}{\gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\eta t} \int_t^\infty \theta e^{-\theta u} e^{-\eta(u-t)} du dt$$
$$= \frac{\eta^\alpha}{\gamma(\alpha)} \frac{\theta}{\theta+\eta} \int_0^\infty t^{\alpha-1} e^{-(\eta+\theta)t} dt = \frac{\theta}{\theta+\eta} \left(\frac{\eta}{\theta+\eta}\right)^\alpha \frac{1}{\gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-ut} du = \frac{\theta}{\theta+\eta} \left(\frac{\eta}{\theta+\eta}\right)^\alpha.$$

And,

$$\mathrm{P}\{V < x, X \leqslant V < X + Y\} = \int_0^x \mathrm{P}\{V < x, t \leqslant V < t + Y\} \frac{\eta^\alpha t^{\alpha-1}}{\gamma(\alpha)} e^{-\eta t} dt = \frac{\eta^\alpha}{\gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-\eta t} \int_t^x \theta e^{-\theta u} e^{-\eta(u-t)} du dt$$
$$= \frac{\eta^\alpha}{\gamma(\alpha)} \frac{\theta}{\theta+\eta} \int_0^x t^{\alpha-1} \left(e^{-(\theta+\eta)t} - e^{-(\theta+\eta)x}\right) dt.$$

Thus, the conditional probability distribution is

$$\mathrm{P}\{V < x | X \leqslant V < X + Y\} = \frac{\mathrm{P}\{V < x, X \leqslant V < X + Y\}}{\mathrm{P}\{X \leqslant V < X + Y\}} = \frac{(\theta+\eta)^\alpha}{\gamma(\alpha)} \int_0^x t^{\alpha-1} \left(e^{-(\theta+\eta)t} - e^{-(\theta+\eta)x}\right) dt.$$

The density function is

$$g\{x | X \leqslant V < X + Y\} = \frac{(\theta+\eta)^{\alpha+1}}{\gamma(\alpha+1)} x^\alpha e^{-(\theta+\eta)x}.$$

Evidently, under the condition $X \leqslant V < X + Y$ V follows Γ-distribution with parameters $\alpha + 1$ and $\theta + \eta$. □

Certainly, If X follows an *Erlang* distribution with parameters n(n is any integer) and η, and V follows an exponential distribution parameter θ, we have the corollaries below.

Corollary 4.1. *If X and V are independent mutually, under the condition X < V, X follows an Erlang distribution with parameters n and* $\theta + \eta$.

Corollary 4.2. *If X and V are independent mutually, under the condition* $X \leqslant V < X + Y$, *V follows an Erlang distribution with parameters n + 1 and* $\theta + \eta$.

4.2. Waiting time distribution

Let W and $W^*(s)$ be the steady-state waiting time and its LST, respectively. Firstly, assume H_0 be the probability that the server stays in the vacation period and the new customer should wait when he (she) arrives, and H_1 be the probability that the server is in the service period and the new customer should wait when he (she) arrives. We can easily compute

$$H_0 = \beta(1-\xi)(1-\sigma)\gamma, \quad H_1 = \beta[(1-\delta)\xi(1-\gamma) + (1-\sigma)\alpha(\xi-\gamma)].$$

Theorem 5. *If* $\rho < 1$ *and* $\theta > 0$, *the LST of the steady-state waiting time W is*

$$\begin{aligned} W^*(s) = 1 - H_0 - H_1 + H_0 \frac{\theta+\eta(1-\gamma)}{\theta+\eta(1-\gamma)+s}\left[p_0 + (1-p_0)\frac{\mu(1-\gamma)}{\mu(1-\gamma)+s}\right] \\ + H_1 \frac{(\mu+s)(1-\xi)}{s+\mu(1-\xi)} \frac{\mu}{\mu+s}\left[p_1 + (1-p_1)\frac{(\mu+s)(1-\gamma)}{\mu(1-\gamma)+s}\right], \end{aligned} \tag{13}$$

where $p_0 = \frac{\eta(1-\gamma)}{\theta+\eta(1-\gamma)}, \quad p_1 = \frac{(1-\delta)\xi(1-\gamma)}{(1-\delta)\xi(1-\gamma)+(1-\sigma)\alpha(\xi-\gamma)}$.

Proof. Firstly, we obtain the probability that a new customer should not wait

$$\mathrm{P}\{W=0\} = \pi_{00} + \pi_{01} = \beta(1-\xi)(1-\gamma)(1-\sigma+1-\delta) = 1 - H_0 - H_1. \tag{14}$$

When a new customer arrives, if there are k customers and the server stays in the normal working period, the waiting time equals the sum of k service times by the rate μ, then we have

$$\begin{aligned} \sum_{k=1}^{\infty} \pi_{k1} W_{k1}^*(s) = \sum_{k=1}^{\infty} \pi_{k1}\left(\frac{\mu}{\mu+s}\right)^k = K\frac{\mu}{s+\mu(1-\xi)}\left[\xi(1-\delta) + (1-\sigma)\alpha(\xi-\gamma)\frac{\mu+s}{s+\mu(1-\gamma)}\right] \\ = H_1 \frac{(\mu+s)(1-\xi)}{s+\mu(1-\xi)} \frac{\mu}{\mu+s}\left[p_1 + (1-p_1)\frac{(\mu+s)(1-\gamma)}{\mu(1-\gamma)+s}\right], \end{aligned} \tag{15}$$

where p_1 is defined as in the theorem.

Next, we consider the situation that there are k customers and the server stays in the vacation period when the new customer arrives. For convenience, denote V and $S_\eta^{(k)}$ the vacation time and the sum of k service times during the vacation, respectively. Evidently, $S_\eta^{(k)}$ follows an *Erlang* distribution with parameters k and η. There are two cases:

Case 1: if the residual vacation time is longer than k service times by the rate η, i.e., $V > S_\eta^{(k)}$, the waiting time is the sum of k service times with the rate η under the condition $V > S_\eta^{(k)}$. From Lemma 4.1, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \pi_{k0}\, \mathrm{P}\left\{V > S_\eta^{(k)}\right\}\left(\frac{\theta+\eta}{\theta+\eta+s}\right)^k = K(1-\sigma)\sum_{k=1}^{\infty}\gamma^k\left(\frac{\eta}{\eta+\theta}\right)^k\left(\frac{\theta+\eta}{\theta+\eta+s}\right)^k = K(1-\sigma)\frac{\eta\gamma}{s+\theta+\eta(1-\gamma)} \\ = H_0\frac{\theta+\eta(1-\gamma)}{s+\theta+\eta(1-\gamma)}p_0. \end{aligned}$$

Case 2: If at the j($j < k$)th services completion instant, the vacation has ended, i.e., $S_\eta^{(j)} \leqslant V < S_\eta^{(j+1)}$, the waiting time equals the sum of the residual vacation time under the condition $S_\eta^{(j)} \leqslant V < S_\eta^{(j+1)}$ plus $k-j$ service times by the rate μ. Thus, it follows from Lemma 4.2 that

$$\begin{aligned} \sum_{k=1}^{\infty}\pi_{k0}\sum_{j=0}^{k-1}\mathrm{P}\left\{S_\eta^{(j)} \leqslant V < S_\eta^{(j+1)}\right\}\left(\frac{\theta+\eta}{\theta+\eta+s}\right)^{j+1}\left(\frac{\mu}{\mu+s}\right)^{k-j} &= K(1-\sigma)\sum_{k=1}^{\infty}\gamma^k\frac{\theta}{\theta+\eta}\left(\frac{\eta}{\theta+\eta}\right)^j\left(\frac{\theta+\eta}{\theta+\eta+s}\right)^{j+1}\left(\frac{\mu}{\mu+s}\right)^{k-j} \\ &= K(1-\sigma)\frac{\theta\gamma}{s+\theta+\eta(1-\gamma)}\frac{\mu}{s+\mu(1-\gamma)} \\ &= H_0\frac{\theta+\eta(1-\gamma)}{s+\theta+\eta(1-\gamma)}(1-p_0)\frac{\mu(1-\gamma)}{s+\mu(1-\gamma)}. \end{aligned}$$

We have

$$\sum_{k=1}^{\infty}\pi_{k0}W_{k0}^{*}(s) = H_0\frac{\theta+\eta(1-\gamma)}{\theta+\eta(1-\gamma)+s}\left[p_0+(1-p_0)\frac{\mu(1-\gamma)}{\mu(1-\gamma)+s}\right]. \quad (16)$$

Further,

$$W^{*}(s) = \pi_{00} + \sum_{k=1}^{\infty}\pi_{k1}W_{k1}^{*}(s) + \sum_{k=1}^{\infty}\pi_{k0}W_{k0}^{*}(s).$$

From (14)–(16), we have the result in Theorem 5. □

With the structure in Theorem 5, we can get the mean waiting time of an arbitrary customer:

$$E(W) = H_0\frac{\theta+\mu(1-\gamma)}{\theta+\eta(1-\gamma)}\frac{1}{\mu(1-\gamma)} + H_1\left[\frac{1}{\mu(1-\xi)} + \frac{(1-\sigma)\alpha(\xi-\gamma)}{(1-\delta)\xi(1-\gamma)+(1-\sigma)\alpha(\xi-\gamma)}\frac{\gamma}{\mu(1-\gamma)}\right].$$

Remark 2. For the Eq. (13), the steady-state waiting time of an arbitrary customer has the special probability explanation. The waiting time equals 0 with the probability $1 - H_0 - H_1$; with the probability H_0, it equals the sum of one exponential random variable with the rate $\theta + \eta(1-\gamma)$ and one modified exponential random variable with the rate $\mu(1-\gamma)$; with the probability H_1, it equals the sum of three randem variables: two exponential random variable with the rates $\mu(1-\xi)$ and μ, respectively, and one modified exponential random variable with the rate $\mu(1-\gamma)$.

Table 1
Steady-state probabilities in the $D/M/1/SWV$ model with parameters: $\lambda = 0.75$, $\mu = 1.5$, $\theta = 0.8$ and $\eta = 0.6$ and with $E(L) = 1.0732$ and $E(W) = 0.3503$.

k	π_{k0}	π_{k1}	$\pi_k(\pi_{k0} + \pi_{k0})$	p_k
0	0.3783	0.2789	0.6571	0.4487
1	0.0675	0.1808	0.2483	0.3707
2	0.0120	0.0589	0.0709	0.1317
3	0.0021	0.0159	0.0181	0.0368
4	3.8275e−004	0.0039	0.0043	0.0093
5	6.8265e−006	9.2587e−004	9.9413e−004	0.0022
6	1.2175e−005	2.1053e−004	2.2271e−004	5.0467e−004
7	2.1715e−006	4.6775e−005	4.8947e−005	1.1271e−004
8	3.8730e−007	1.0217e−005	1.0664e−005	2.4715e−005
9	6.9077e−008	2.2031e−006	2.2722e−006	5.3453e−006
Sum	0.4604	0.5395	0.9999	0.9999

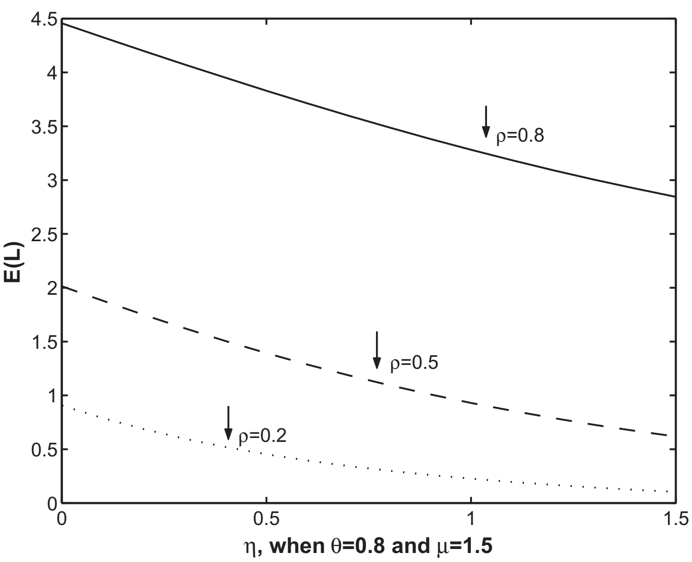

Fig. 1. The Mean queue length $E(W)$ with the change of η in the E_2/M/1 model under three cases for ρ.

5. Numerical examples

In this section, we consider a couple of sample numerical examples: the D/M/1/SWV and E_2/M/1/SWV queues, where D stands for a deterministic arrival interval and E_2 stands for a two-phase Erlang distribution.

For the D/M/1/SWV queue, we use a table (Table 1) to present the steady-state queue length distributions π_{k0}, π_{k1} and $p_k(k \geqslant 0)$ at the arrival and arbitrary epochs under the fixed parameters. Evidently, when k achieves a special value, the probabilities become smaller and can even be omitted in some extent. For example, in Table 1, when $k = 9$, π_k and p_k become so smaller that the probabilities that there are more than k customers approach to zero. And, the sums of π_k and p_k from 0 to 9 also verify this result and it is demonstrated that our method and results are valuable and can be applied to obtain the system probability directly. Certainly, from the distributions thus obtained, important performance measures of primal interest, such as the mean queue length $E(L)$ and the mean waiting time $E(W)$, can be obtained and shown in Table 1.

In the second example of the $E_2/M/1/SWV$ queue, the steady-state queue length distributions can be obtained in the similar way, but now we focus on the effects of system parameters on the performance measures of primal interest, such as $E(L)$, $E(W)$ and state probabilities of the server. Assume the normal service rate $\mu = 1.5$ being fixed. First, we use the following in-

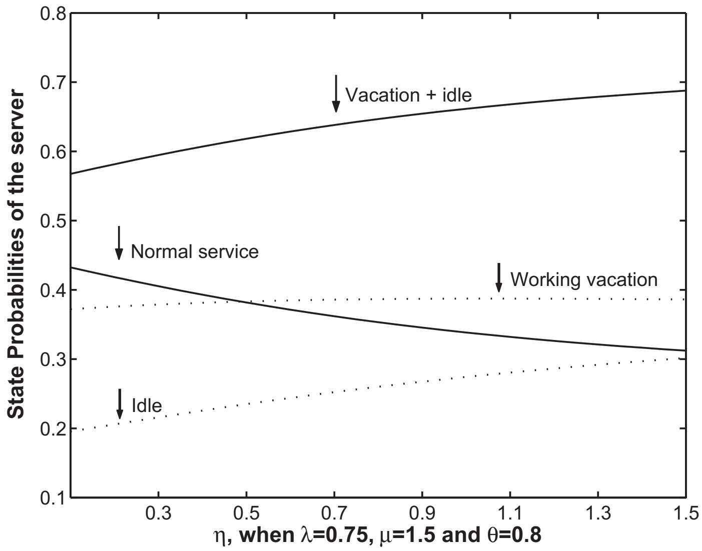

Fig. 2. The State probabilities of the server with the change of η in the E_2/M/1 model.

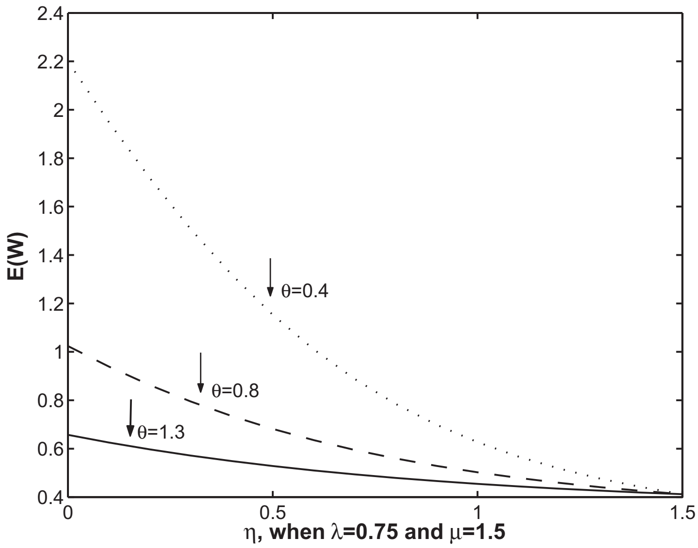

Fig. 3. The waiting time $E(W)$ with the change of η in the E_2/M/1 model under three cases for θ.

put parameters: the arrival rates λ are 0.3, 0.75, and 1.2, representing light, medium, and high traffic intensities, namely, $\rho = \lambda/\mu = 0.2$, 0.5, and 0.8, respectively. And assume θ with a fixed value 0.8. In Fig. 1, we plot the change trend of $E(L)$ as η increases from 0 to 1.5 under the corresponding three traffic intensities. It is obvious that, if η is fixed, the higher ρ is, the larger the mean queue length $E(L)$ becomes. We also see that increased η leads to the smaller mean queue length $E(L)$ for any given traffic intensity ρ. It demonstrates that the working vacation policy can decrease the number of the waiting customers in the system and enhance the system efficiency.

In Fig. 2, we illustrate the state probabilities of the server in the $E_2/M/1/SWV$ with parameters $\lambda = 0.75$, $\mu = 1.5$ and $\theta = 0.8$. Evidently, with the increase of η, the probabilities that the server stays in the working vacation and idle period get larger and the opposite state (normal service) probability declines. This phenomena can be explained from the simple busy analysis. In the single vacation system, the duration of a vacation period (V) is fixed in some sense and when the working vacation service rate increases, a whole cycle period (the vacation period plus the next idle period plus the next normal service period) will decrease. To some extent, the probability that the server stays in the working vacation can be seen as the ratio of the vacation period to the whole cycle period, then the change trends of these state probabilities also explain what happens.

From a practical perspective, a very useful measure is the waiting time of an arbitrary customer, thus we pay attention to the change trends of $E(W)$ for different parameters in the $E_2/M/1/SWV$ queue below. In Fig. 3, we plot the corresponding $E(W)$

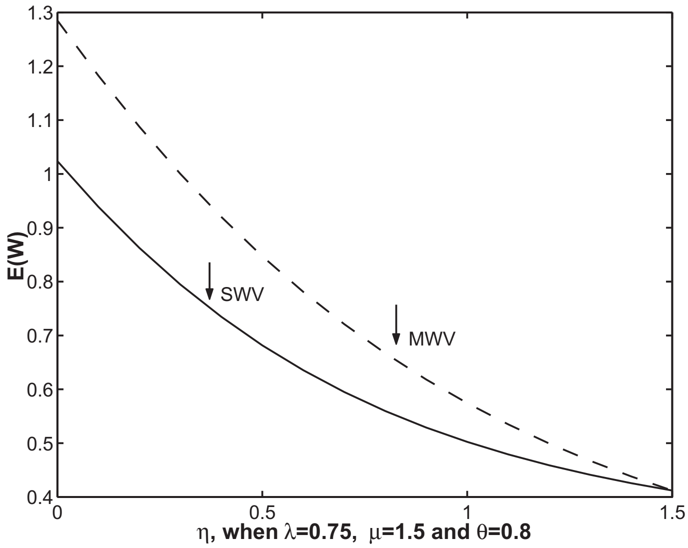

Fig. 4. The waiting time $E(W)$ with the change of η in $E_2/M/1$ models under MWV and SWV policies.

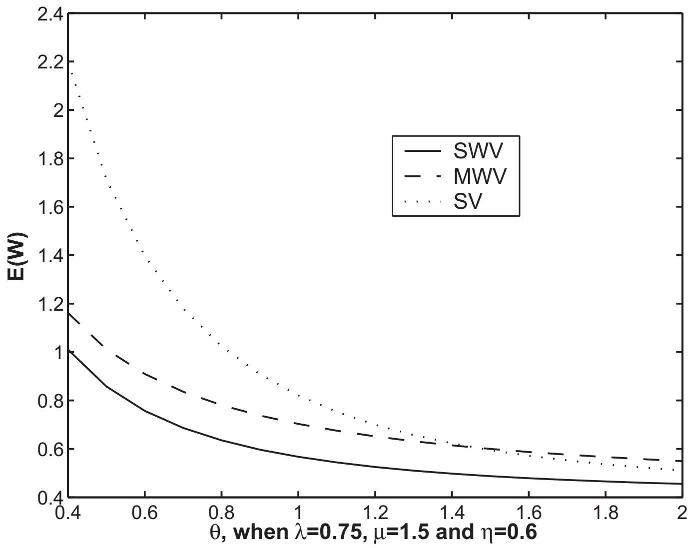

Fig. 5. The waiting time $E(W)$ with the change of θ in $E_2/M/1$ models under three policies.

for $\theta = 0.4$, 0.8 and 1.3. Similar to the mean queue length, $E(W)$ gets smaller as η increases. Meanwhile, as illustrated in the figure, the larger vacation rate shows the smaller waiting time if the value of η is fixed. Note that the mean waiting time for $\eta = 1.5$ corresponds to the case without vacation and θ has no influence on $E(W)$.

To demonstrate some advantages of the single working vacation policy, in Figs. 4 and 5, we present some comparison results. Fig. 4 shows the comparison between the model with the multiple working vacations (MWV) considered in Baba [1] and our model with the SWV. As demonstrated in Fig. 4, the single working vacation policy can make the waiting time decrease correspondingly. Finally, in Fig. 5, we add the classic single vacation, denoted as SV, under which the server can not work during the vacation period, to the comparison and consider the effects of θ rather than η. Evidently, no matter under which case the system is, the mean waiting time becomes smaller as the vacation time θ^{-1} decreases. It is obvious that the waiting time of an arbitrary customer in our model with the SWV is the smallest if θ is fixed. Although, we only focus on the comparison of the mean waiting time, but we believe that the similar results exist for other performance measures as we show above. It explains again that the model with the single working vacation has certain practical implication for the various systems.

6. Conclusions

In this paper, our main goal is to establish the theoretical foundations for applications and obtain the specific computation expressions for the performance measures, such as the mean queue length, mean waiting time and state probabilities of the server. Such results can be used to simulate the practical problems directly. We have done work in several aspects:

1. Parallel to the multiple working vacations in Baba [1], we establish a new model with the single working vacation and using the matrix analytic method, the queue length distribution and mean number of customers are obtained.
2. We present the stochastic decomposition structures of the system measures in such a model, which demonstrate the probability relation with the classical vacation queues.
3. Another important contribution of our study is to use the the closed property of conditional probability for Γ-distributions to analyze the waiting time of an arbitrary customer. The proof of such a property is well known, and it is possible to be used to analyze other models.
4. We also present some numerical examples which demonstrate that the theoretical results obtained above are reasonable and can be applied to solve the practical problems directly.

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