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The discrete-time GI/Geo/1 queue with working vacations and vacation interruption

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Abstract

In this paper, we consider a GI/Geo/1 queue with working vacations and vacation interruption. The server takes the original work at the lower rate rather than completely stopping during the vacation period. Meanwhile, we introduce vacation interruption policy: the server can come back to the normal working level once there are customers after a service completion during the vacation period, thus the server may not accomplish a complete vacation. Using matrix-geometric solution method, we obtain the steady-state distributions for the number of customers in the system at arrival epochs, and waiting time for an arbitrary customer. Meanwhile, we explain the stochastic decomposition properties of queue length and waiting time.

Keywords: Discrete-time queue; Vacation interruption; Working vacations; Matrix-geometric approach; Stochastic decomposition

1. Introduction

In this paper, we will consider a discrete-time queue that the customer arrivals and service completions occur at the discrete-time instants. The continuous time queues with vacations have been well investigated. Details can be seen in the surveys of Doshi [1,2] and the monographs of Takagi [3] and Tian and Zhang [4]. But, discrete-time queues are more suitable to model and analyze the digital communication system. Meisling [5] firstly presented the discrete-time queue. Hunter [6] collected the early results of the discrete-time queues. Takagi [7] gave the analysis of various kinds of Geo/G/1 queues. Tian and Zhang [8], Zhang and Tian [9] investigated a GI/Geo/1 with multiple vacations and a Geo/G/1 queue with multiple adaptive vacations, respectively. Alfa [10] systematically analyzed a series of models with non-exhaustive service in which both vacation time and service time follow phase type distribution.

In this paper, we will consider a GI/Geo/1 queue with vacation interruption and working vacations. For GI/M/1 type queues with server vacations, Tian et al. [11] used the matrix geometric solution method to

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analyze and obtained the expressions of the rate matrix and proved the stochastic decomposition properties for queue length and waiting time in a GI/M/1 vacation model with multiple exponential vacations. Independently, Chatterjee and Mukherjee [12] also researched the GI/M/1 with server vacations. Subsequently, Tian and Zhang [13,8] discussed the GI/M/1 queue with PH vacations or setup times and discrete-time GI/Geo/1 queue with server vacations.

In the queueing models with vacations in all these literatures, the server cannot take the original work and can take some assistant work during the vacation period. Recently, Servi and Finn [14] first studied an M/M/1 queue with working vacation: a customer is served at a lower rate during a vacation. Their work is motivated and illustrated by the analysis of a WDM optical access network using multiple wavelength which can be reconfigured. Subsequently, Kim et al. [15], Wu and Takagi [16] generalized results in Servi and Finn [14] to an M/G/1 queue with working vacation. Baba [17] extended this study to a GI/M/1 queue with working vacation by the matrix-analysis method. Li et al. [18] also analyzed the discrete-time GI/Geo/1 queue with working vacations and gave the stochastic results for the queue length and waiting time.

Vacation interruption is introduced by the authors in this paper: the server can stop the vacation once some indices of the system, such as the number of customers, achieve the certain value in the vacation period. Certainly, it is possible for the server to take an interrupted vacation, so we call this policy vacation interruption. In some practical situations, some urgent things happen in the server's vacation period and the server must come back to work rather than continuing to take the residual vacation. Especially, for working vacation models, the server can take service in the vacation period and must come back in some situations, for example when the number of customers exceeds the special value and if the server continues to take the vacation, the costs of waiting customers and working in the vacation period will be large. So, vacation interruption is more reasonable to the queueing system with vacations.

In this paper, the working vacation and vacation interruption are connected and the server enters into vacation when there are no customers and he can take service at the lower rate in the vacation period. If there are customers in the system after a service is completed in the vacation period, the server will come back to the normal working level. Otherwise, he continues the vacation. Using matrix-geometric approach, we give concise expressions of distributions for queue length at an arrival epoch. Meanwhile, we also get the steady-state distribution for the waiting time of an arbitrary customer. Furthermore, we also present the stochastic decomposition results for queue length and waiting time.

The rest of this paper is organized as follows. In Section 2, we firstly provide the model formulation of GI/Geo/1 with vacation interruption and multiple working vacations and establish the embedded Markov chain at the arrival epochs. In Section 3, we give steady-state distribution for the embedded Markov chain at arrival epochs. In Section 4, the distribution for the waiting time of an arbitrary customer is obtained. Lastly, we explain the stochastic decomposition property for queue length at the arrival epoch and two conditional stochastic decomposition structures for waiting time in Section 5.

2. Model formulation and embedded Markov chain

Assume that customer arrivals occur at discrete-time instants $t = n^-$, $n = 0, 1, \ldots$ Inter-arrival times are independent and identical distributed (i.i.d) sequences with a general discrete distribution $\{\lambda_j, j \geqslant 1\}$ with mean and probability generating function (PGF) $1/\lambda$ and $A(z)$, respectively.

The beginning and ending of service occur at discrete-time instants $t = n^+$, $n = 0, 1, \ldots$ The service times in a regular busy period and working vacation period follow geometric distributions with the parameters μ, $\eta(\eta < \mu)$, respectively.

A server begins a working vacation at the instant when the queue becomes empty. Suppose the beginning and ending of vacation occur at the instant $t = n^+$. Vacation time V follows geometric distributions with the parameters θ. During a working vacation arriving customers are served according to arrival order by the rate η; when a service completes during the vacation period, if there are customer in the queue, the server switches service rate from η to μ rather than keeping on the vacation, and a regular busy period starts. Otherwise, the server keeps on vacation. Meanwhile, when the vacation ends, if there are customers, the server also comes to the normal working level. Otherwise, he continues another vacation.

Suppose τ_n be the arrival epoch of nth customers with $\tau_0 = 0$. Let $L_n = L(\tau_n - 0)$ be the number of the customers before the nth arrival. Define

$$J_n = \begin{cases} 1, & \text{the nth arrival occurs during a service period,} \\ 0, & \text{the nth arrival occurs during a working vacation period.} \end{cases}$$

The process $\{(L_n, J_n), n \geqslant 1\}$ is a Markov chain with the state space

$$\Omega = \{(0,0)\} \bigcup \{(k,j),\ k \geqslant 1,\ j = 0,1\}.$$

In order to express the transition matrix of (L_n, J_n), let

$$p_{(i,j),(k,l)} = P(L_{n+1} = k, J_{n+1} = l | L_n = i, J_n = j).$$

Meanwhile, we introduce the expressions below

$$a_j = \sum_{r=j}^{+\infty} \lambda_r \binom{r}{j} \mu^j (1-\mu)^{r-j}, \quad j \geqslant 0;$$

$$b_j = \sum_{r=j}^{+\infty} \lambda_r \sum_{i=1}^{r-(j-1)} \eta(1-\eta)^{i-1}(1-\theta)^{i-1} \binom{r-i}{j-1} \mu^j (1-\mu)^{r-i-j+1}, \quad j \geqslant 1;$$

$$c_j = \sum_{r=j+1}^{+\infty} \lambda_r \sum_{i=1}^{r-j} \theta(1-\theta)^{i-1}(1-\eta)^i \binom{r-i}{j} \mu^j (1-\mu)^{r-i-j}, \quad j \geqslant 0.$$

where $a_j, j \geqslant 0$ represent the probability that there are j service completions during an inter-arrival time in the busy period; $b_j, j \geqslant 0$ represent the probability that, during an inter-arrival time there are a service completion in the vacation time and $j-1$ service completions in the busy period; $c_j, j \geqslant 0$ represent the probability that no customer is served by rate η, and there are j service completions by the rate η during an inter-arrival time.

Now, we consider the transition probabilities of (L_n, J_n). First, the transition from $(i,1)$ to $(j,1)$ occurs if $i+1-j$ services complete during an inter-arrival time. Therefore, we have

$$p_{(i,1),(j,1)} = a_{i+1-j}, \quad i \geqslant 1,\ 1 \leqslant j \leqslant i+1.$$

Second, the transition from $(i,0)$ to $(j,0)$ occurs only if $j = i+1$. So, we have

$$p_{(i,0),(j,0)} = \sum_{r=1}^{+\infty} \lambda_r (1-\theta)^r (1-\eta)^r = A((1-\theta)(1-\eta)), \quad j = i+1.$$

Third, there are two possible cases to make the transition from $(i,0)$ to $(j,1)$. Case 1: if the residual vacation time is larger than or equal to the service time in the vacation period, during an inter-arrival time, there is one service completion by the rate η, then the server switches to the normal rate μ and $i-j$ customers are served by the rate μ; Case 2: if the residual vacation time is shorter than the service time in the vacation period, no customer is served during the vacation time and $i+1-j$ customers are served during the service period. Then, we have

$$p_{(i,0),(j,1)} = \begin{cases} b_{i+1-j} + c_{i+1-j} & 1 \leqslant j \leqslant i, \\ \sum_{r=1}^{+\infty} \lambda_r \sum_{i=1}^{r} (1-\theta)^{i-1}\theta(1-\eta)^i(1-\mu)^{r-i} = c_0, & j = i+1. \end{cases}$$

Similarly,

$$\begin{aligned} p_{(i,0),(0,0)} &= \sum_{j=i}^{+\infty} b_{j+1} + \sum_{j=i+1}^{+\infty} c_j = \sum_{j=0}^{+\infty} b_{j+1} + \sum_{j=0}^{+\infty} c_j - \sum_{j=1}^{i} (c_j + b_j) - c_0 \\ &= 1 - A((1-\theta)(1-\eta)) - \sum_{j=1}^{i} (c_j + b_j) - c_0, \quad i \geqslant 1. \end{aligned}$$

In the above equation, we can compute

$$\begin{aligned}\sum_{j=0}^{+\infty} b_{j+1} &= \sum_{j=1}^{+\infty}\sum_{r=j}^{+\infty}\lambda_r\sum_{i=1}^{r-(j-1)}\eta(1-\eta)^{i-1}(1-\theta)^{i-1}\binom{r-i}{j-1}\mu^j(1-\mu)^{r-i-j+1}\\ &= \sum_{r=1}^{+\infty}\lambda_r\sum_{i=1}^{r}\eta(1-\eta)^{i-1}(1-\theta)^{i-1}\sum_{j=1}^{r-(i-1)}\binom{r-i}{j-1}\mu^j(1-\mu)^{r-i-j+1}\\ &= \eta\sum_{r=1}^{+\infty}\lambda_r\sum_{i=1}^{r}(1-\eta)^{i-1}(1-\theta)^{i-1} = \frac{\eta}{1-(1-\theta)(1-\eta)}(1-A((1-\theta)(1-\eta))).\end{aligned}$$

Similarly,

$$\sum_{j=0}^{+\infty} c_j = \frac{\theta(1-\eta)}{1-(1-\theta)(1-\eta)}(1-A((1-\theta)(1-\eta))).$$

From the above equations, we obtain

$$p_{(0,0),(0,0)} = 1-A((1-\theta)(1-\eta))-c_0,\quad p_{(i,1),(0,0)} = 1-\sum_{k=0}^{i}a_k,\qquad i\geqslant 1.$$

Using the lexicographical sequence for the states, the transition probability matrix of (L_n, J_n) can be written as the Block–Jacobi matrix

$$\widetilde{P} = \begin{bmatrix} B_{00} & A_{01} & & & & \\ B_1 & A_1 & A_0 & & & \\ B_2 & A_2 & A_1 & A_0 & & \\ B_3 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

where

$$\begin{aligned} B_{00} &= 1-A((1-\theta)(1-\eta))-c_0;\quad A_{01} = (A((1-\theta)(1-\eta)), c_0);\\ A_0 &= \begin{bmatrix} A((1-\theta)(1-\eta)) & c_0\\ 0 & a_0\end{bmatrix};\quad A_k = \begin{bmatrix} 0 & b_k+c_k\\ 0 & a_k\end{bmatrix},\quad k\geqslant 1;\\ B_k &= \begin{bmatrix} 1-A((1-\theta)(1-\eta))-\sum_{i=1}^{k}(c_i+b_i)-c_0\\ 1-\sum_{i=0}^{k}a_i\end{bmatrix},\quad k\geqslant 1.\end{aligned}$$

The structure of $\widetilde{P}$ indicates that (L_n, J_n) is irreducible and aperiodic.

3. Steady-state distribution at arrival epochs

Then, we derive the steady-state distribution at arrival epochs by using Neuts' matrix-geometric approach. In order to derive the steady-state distribution, we need the lemma below.

Lemma 1. *If $\rho = \lambda/\mu < 1$ and $\theta > 0$, then the matrix equation $R = \sum_{k=0}^{\infty}R^kA_k$ has the minimal non-negative solution*

$$R = \begin{bmatrix}\gamma & \alpha(\gamma-\xi)\\ 0 & \xi\end{bmatrix},$$

where ξ *is the unique root in the range* $0 < z < 1$ *of the equation* $z = A(1 - \mu(1 - z))$ *and*

$$\gamma = A((1-\theta)(1-\eta)), \quad \alpha = \frac{\theta(1-\eta)+\eta\gamma}{(1-\theta)(1-\eta)-(1-\mu(1-\gamma))}.$$

Proof. Because all A_k are upper triangular, we can assume that R has the same structure as

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}.$$

Then, for $k \geqslant 1$, we have

$$R^k = \begin{bmatrix} r_{11}^k & r_{12}\sum_{j=0}^{k-1} r_{11}^j r_{22}^{k-1-j} \\ 0 & r_{22}^k \end{bmatrix}.$$

Substituting R^k into the matrix equation, we obtain

$$\begin{cases} r_{11} = A((1-\theta)(1-\eta)), \\ r_{12} = c_0 + \sum_{k=1}^{+\infty} r_{11}^k (b_k + c_k) + r_{12}\sum_{k=1}^{+\infty} a_k \sum_{j=0}^{k-1} r_{11}^j r_{22}^{k-1-j}, \\ r_{22} = \sum_{k=0}^{+\infty} a_k r_{22}^k = A(1-\mu(1-r_{22})). \end{cases} \tag{2}$$

As it is well known, if $\rho = \lambda/\mu < 1$, the third equation has the unique root $r_{22} = \xi$ in the range $0 < r_{22} < 1$. Taking $r_{22} = \xi$ and $r_{11} = \gamma$ into the second equation, we have

$$r_{12}\left(1 - \sum_{k=1}^{+\infty} a_k \sum_{j=0}^{k-1} r_{11}^j r_{22}^{k-1-j}\right) = \sum_{k=1}^{+\infty} r_{11}^k b_k + \sum_{k=0}^{+\infty} r_{11}^k c_k \tag{3}$$

and, we can compute

$$\begin{aligned} \sum_{k=1}^{+\infty} r_{11}^k b_k &= \sum_{k=1}^{+\infty} \gamma^k \sum_{r=k}^{+\infty} \lambda_r \sum_{i=1}^{r-(k-1)} \eta(1-\eta)^{i-1}(1-\theta)^{i-1}\binom{r-i}{k-1}\mu^{k-1}(1-\mu)^{r-i-k+1} \\ &= \gamma \sum_{r=1}^{+\infty} \lambda_r \sum_{i=1}^{r} \eta(1-\eta)^{i-1}(1-\theta)^{i-1} \sum_{k=1}^{r-i+1}\binom{r-i}{k-1}(\mu\gamma)^{k-1}(1-\mu)^{r-i-k+1} \\ &= \eta\gamma \sum_{r=1}^{+\infty} \lambda_r \sum_{i=1}^{r} (1-\eta)^{i-1}(1-\theta)^{i-1}(1-\mu(1-\gamma))^{r-i} \\ &= \frac{\eta\gamma}{(1-\theta)(1-\eta)-(1-\mu(1-\gamma))}(A((1-\theta)(1-\eta)) - A(1-\mu(1-\gamma))). \end{aligned}$$

Similarly,

$$\sum_{k=0}^{+\infty} r_{11}^k c_k = \frac{\theta(1-\eta)}{(1-\theta)(1-\eta)-(1-\mu(1-\gamma))}(A((1-\theta)(1-\eta)) - A(1-\mu(1-\gamma))).$$

Thus, we have

$$\sum_{k=1}^{+\infty} r_{11}^k b_k + \sum_{k=0}^{+\infty} r_{11}^k c_k = \alpha(\gamma - A(1-\mu(1-\gamma)))$$

and, we easily have

$$1 - \sum_{k=1}^{+\infty} a_k \sum_{j=0}^{k-1} r_{11}^j r_{22}^{k-1-j} = 1 - \frac{A(1-\mu(1-\gamma)) - \xi}{\gamma - \xi} = \frac{\gamma - A(1-\mu(1-\gamma))}{\gamma - \xi}.$$

Substituting the above equations into (3), we finally obtain the expression for R. We have that$\alpha(\gamma-\xi)>0$ for $\rho<1$ and $\theta>0$. The specific proof is similar with Tian and Zhang [8] and we do not explain here. □

Theorem 1. *The Markov chain* $\widetilde{P}$ *is positive recurrent if and only if* $\rho<1$ *and* $\theta>0$.

Proof. Based on Theorem 1.5.1 of Neuts [19], the Markov chain $\widetilde{P}$ is positive recurrent if and only if the spectral radius $\mathrm{SP}(R)=\max\{\gamma,\xi\}$ of the rate matrix R is less than 1, and the matrix

$$B[R]=\begin{bmatrix} B_{00} & A_{01} \\ \sum_{k=1}^{+\infty} R^{k-1}B_k & \sum_{k=1}^{+\infty} R^{k-1}A_k \end{bmatrix}$$

has a positive left invariant vector. Evidently, $\mathrm{SP}(R)=\max\{\gamma,\xi\}<1$. Substituting the expressions for R, A_k and B_k in $B[R]$, we obtain

$$B[R]=\begin{bmatrix} 1-\gamma-c_0 & \gamma & c_0 \\ 1-\frac{a_0\alpha(\gamma-\xi)}{\gamma\xi}+\frac{c_0}{\gamma} & 0 & \frac{a_0\alpha(\gamma-\xi)}{\gamma\xi}-\frac{c_0}{\gamma} \\ \frac{a_0}{\xi} & 0 & 1-\frac{a_0}{\xi} \end{bmatrix}.$$

It can be easily verify that $B[R]$ has the left invariant vector

$$K(1,\gamma,\alpha(\gamma-\xi)). \tag{4}$$

Thus, if $\rho<1$ and $\theta>0$, the Markov chain $\widetilde{P}$ is positive recurrent. □

If $\rho<1$, let (L,J) be the stationary limit of the QBD process $\{L_n,J_n\}$. Let

$$\pi_0=\pi_{00};\quad \pi_k=(\pi_{k0},\pi_{k1}),\qquad k\geqslant 1,$$
$$\pi_{kj}=P\{L=k,J=j\}=\lim_{n\to\infty}P\{L_n=k,J_n=j\},\quad (k,j)\in\Omega.$$

Theorem 2. *If* $\rho<1$, *the stationary probability distribution of* (L,J) *is*

$$\begin{cases} \pi_{k0}=(1-\xi)\sigma\gamma^k, & k\geqslant 0, \\ \pi_{k1}=(1-\xi)\sigma\alpha(\gamma-\xi)\sum_{j=0}^{k-1}\xi^j\gamma^{k-1-j}, & k\geqslant 1, \end{cases} \tag{5}$$

where

$$\sigma=\frac{1-\gamma}{1-\xi+\alpha(\gamma-\xi)},\quad \gamma=A((1-\theta)(1-\eta)).$$

Proof. With Theorem 1.5.1 of Neuts [19], $(\pi_{00},\pi_{10},\pi_{11})$ is given by the positive left invariant vector (4) and satisfies the normalizing condition

$$\pi_{00}+(\pi_{10},\pi_{11})(I-R)^{-1}e=1$$

and, substituting R into the above relation, we easily get

$$K=(1-\xi)\frac{1-\gamma}{1-\xi+\alpha(\gamma-\xi)}=(1-\xi)\sigma.$$

Therefore, we obtain

$$(\pi_{10},\pi_{11})=(1-\xi)\sigma(\gamma,\alpha(\gamma-\xi)).$$

Using Theorem 1.5.1 of Neuts [19], we have

$$\pi_k=(\pi_{k0},\pi_{k1})=(\pi_{10},\pi_{11})R^{k-1},\quad k\geqslant 1. \tag{6}$$

Taking (π_{10},π_{11}) and R^{k-1} into (6), we easily obtain the theorem. □

Thus, we easily obtain the state probabilities of a server in steady-state.

$$p\{J=0\}=\sum_{k=0}^{+\infty}\pi_{k0}=\frac{1-\xi}{1-\xi+\alpha(\gamma-\xi)},$$
$$p\{J=1\}=\sum_{k=1}^{+\infty}\pi_{k1}=\frac{\alpha(\gamma-\xi)}{1-\xi+\alpha(\gamma-\xi)}. \tag{7}$$

Remark 1. If $\eta = 0$, i.e., the server does not take service in the vacation period, the results are the same with the classical results in the GI/Geo/1 queue with multiple vacations (see [8]).

4. Waiting time for an arbitrary customer

In this section, we assume that the service discipline is first-come first-served. Let W and $W(z)$ be the steady-state waiting time and its PGF, respectively.

Firstly, let H_0 be the probability that the server is in the service period when a new customer arrives, and H_1 be the probability that the server is in the vacation period and the new customer should wait when a new customer arrives. We can easily compute

$$H_0=\frac{\alpha(\gamma-\xi)}{1-\xi+\alpha(\gamma-\xi)},\quad H_1=\frac{(1-\xi)\gamma(1-\eta(1-\gamma))}{1-\xi+\alpha(\gamma-\xi)}.$$

To obtain the distribution for the waiting time, we firstly have two lemmas below. Note that U^* and V represent the service time by the rate η and the vacation time, respectively. Then we have

Lemma 2.1. *If U^* and V are independent mutually, under the condition $U^* \leqslant V$, U^* also follows the geometric distribution with parameters $\theta + \eta(1 - \theta)$.*

Proof. Firstly, we easily compute

$$P\{U^*\leqslant V\}=\sum_{k=1}^{+\infty}(1-\eta)^{k-1}\eta(1-\theta)^k=\frac{\eta(1-\theta)}{\theta+\eta(1-\theta)}.$$

Thus,

$$P\{U^*=k|U^*\leqslant V\}=\frac{P\{U^*=k,U^*\leqslant V\}}{P\{U^*\leqslant V\}}=[(1-\theta)(1-\eta)]^{k-1}[1-(1-\theta)(1-\eta)].\qquad\square$$

Similarly, we have

Lemma 2.2. *If U^* and V are independent mutually, under the condition $U^* > V$, V also follows the geometric distribution with parameters $\theta + \eta(1 - \theta)$, and the distribution is given by*

$$P\{V=k|U^*>V\}=[(1-\theta)(1-\eta)]^k[1-(1-\theta)(1-\eta)],\quad k\geqslant 0.$$

With these lemmas, we can obtain the distribution for waiting time.

Theorem 3. *If $\rho < 1$ and $\theta > 0$, the PGF of stationary waiting time W is*

$$\begin{aligned}W(z)=&1-H_0-H_1+H_0\frac{(1-\xi)(1-(1-\mu)z)}{1-(1-\mu(1-\xi))z}\frac{\mu(1-\gamma)}{1-(1-\mu(1-\gamma))z}+H_1\frac{(1-\gamma)(1-(1-\mu)z)}{1-(1-\mu(1-\gamma))z}\\&\times\left\{\frac{\eta\gamma}{1-\eta(1-\gamma)}\frac{\mu z}{1-(1-\mu)z}+\frac{1-\eta}{1-\eta(1-\gamma)}\frac{(1-(1-\theta)(1-\eta))z}{1-(1-\theta)(1-\eta)z}\right.\\&\left.\times\left(\frac{\eta(1-\theta)}{\theta+\eta(1-\theta)}+\frac{\theta}{\theta+\eta(1-\theta)}\frac{\mu}{1-(1-\mu)z}\right)\right\}.\end{aligned}\tag{8}$$

Proof. Firstly, we easily obtain that the probability that a new customer should not wait is

$$p\{W=0\}=\pi_{00}+\eta\pi_{10}=(1-\xi)\sigma(1+\eta\gamma)=1-H_0-H_1. \tag{9}$$

When a new customer arrives, if there are k customers and the server is in the normal working period, the waiting time equals k service times by the rate μ. Then, we easily have

$$\begin{aligned}\sum_{k=1}^{+\infty}\pi_{k1}W_{k1}(z)&=\sum_{k=1}^{+\infty}\pi_{k1}\left\{\mu\left(\frac{\mu z}{1-(1-\mu)z}\right)^{k-1}+(1-\mu)\left(\frac{\mu z}{1-(1-\mu)z}\right)^{k}\right\}\\&=\frac{\alpha(\gamma-\xi)}{1-\xi+\alpha(\gamma-\xi)}\frac{(1-\xi)(1-(1-\mu)z)}{1-(1-\mu(1-\xi))z}\frac{\mu(1-\gamma)}{1-(1-\mu(1-\gamma))z}\\&=H_0\frac{(1-\xi)(1-(1-\mu)z)}{1-(1-\mu(1-\xi))z}\frac{\mu(1-\gamma)}{1-(1-\mu(1-\gamma))z}.\end{aligned} \tag{10}$$

Then, we consider the situation that there are k customers and the server is in the vacation period when the new customer arrives. There are two cases.

Case 1: if a service completes at the instant $t=n^+$, the waiting time equals $k-1$ service times by the rate μ,

$$\begin{aligned}\eta\sum_{k=2}^{+\infty}\pi_{k0}\left(\frac{\mu z}{1-(1-\mu)z}\right)^{k-1}&=\eta\gamma\frac{(1-\xi)\gamma}{1-\xi+\alpha(\gamma-\xi)}\frac{\mu(1-\gamma)}{1-(1-\mu(1-\gamma))z}\\&=H_1\frac{\eta\gamma}{1-\eta(1-\gamma)}\frac{\mu(1-\gamma)}{1-(1-\mu(1-\gamma))z}.\end{aligned} \tag{11}$$

Case 2: if no service completes at the instant $t=n^+$, there are also two cases.

Firstly, if the residual vacation time is larger than one service time by the rate η, i.e., $V\geqslant U^*$, after a service completion in the vacation period, vacation interruption happens and the server comes back to the normal working level rather than keeping on the vacation. Thus, the waiting time is the sum of one service time by the rate η under the condition $V\geqslant U^*$ and $k-1$ service times by the rate μ. With Lemma 2.1, we have

$$\begin{aligned}&(1-\eta)\sum_{k=1}^{+\infty}\pi_{k0}p\{V\geqslant U^*\}W_{k0}(z)\\&=(1-\eta)(1-\xi)\sigma\sum_{k=1}^{+\infty}\gamma^k\frac{\eta(1-\theta)}{\theta+\eta(1-\theta)}\frac{(1-(1-\theta)(1-\eta))z}{1-(1-\theta)(1-\eta)z}\left(\frac{\mu z}{1-(1-\mu)z}\right)^{k-1}\\&=\frac{(1-\eta)(1-\xi)\gamma}{1-\xi+\alpha(\gamma-\xi)}\frac{\eta(1-\theta)}{\theta+\eta(1-\theta)}\frac{(1-(1-\theta)(1-\eta))z}{1-(1-\theta)(1-\eta)z}\frac{(1-\gamma)(1-(1-\mu)z)}{1-(1-\mu(1-\gamma))z}\\&=H_1\frac{1-\eta}{1-\eta(1-\gamma)}\frac{\eta(1-\theta)}{\theta+\eta(1-\theta)}\frac{(1-(1-\theta)(1-\eta))z}{1-(1-\theta)(1-\eta)z}\frac{(1-\gamma)(1-(1-\mu)z)}{1-(1-\mu(1-\gamma))z}.\end{aligned} \tag{12}$$

Secondly, if $V<U^*$, when a vacation ends, the server does not complete a service during the vacation and comes back to the normal level. Therefore, the waiting time equals the sum of the residual vacation time under the condition $V<U^*$ and k service times by the rate μ. From Lemma 2.2, we obtain

$$\begin{aligned}&(1-\eta)\sum_{k=1}^{+\infty}\pi_{k0}p\{U^*>V\}\widetilde{W}_{k0}(z)\\&=(1-\eta)(1-\xi)\sigma\sum_{k=1}^{+\infty}\gamma^k\frac{\theta}{\theta+\eta(1-\theta)}\frac{1-(1-\theta)(1-\eta)}{1-(1-\theta)(1-\eta)z}\left(\frac{\mu z}{1-(1-\mu)z}\right)^{k}\\&=\frac{(1-\eta)(1-\xi)\gamma}{1-\xi+\alpha(\gamma-\xi)}\frac{\theta}{\theta+\eta(1-\theta)}\frac{1-(1-\theta)(1-\eta)}{1-(1-\theta)(1-\eta)z}\frac{\mu(1-\gamma)z}{1-(1-\mu(1-\gamma))z}\\&=H_1\frac{1-\eta}{1-\eta(1-\gamma)}\frac{\theta}{\theta+\eta(1-\theta)}\frac{1-(1-\theta)(1-\eta)}{1-(1-\theta)(1-\eta)z}\frac{\mu(1-\gamma)z}{1-(1-\mu(1-\gamma))z}.\end{aligned} \tag{13}$$

From (9)–(13), we have the result in Theorem 3 by some algebraic manipulation. □

With the structure in Theorem 3, we can easily get expected waiting time

$$E(W) = H_0\left\{\frac{\xi}{\mu(1-\xi)} + \frac{\mu(1-\gamma)}{1-\mu(1-\gamma)}\right\} + H_1\left\{\frac{\gamma}{\mu(1-\gamma)} + \frac{\eta\gamma}{1-\eta(1-\gamma)}\frac{1}{\mu} + \frac{1-\eta}{1-\eta(1-\gamma)}\frac{1}{\theta+\eta(1-\theta)}\left(1+\theta\frac{1}{\mu}\right)\right\}.$$

Remark 2. Because it is possible that the vacation ends at the instant $t = n^+$ when the new customer arrives at $t = n^-$, the vacation time can be 0 when we consider the waiting time.

5. Stochastic decomposition results

We know that the classical vacation queues have the stochastic decomposition structures for queue length and waiting time. The model in this paper also has this property.

Firstly, we discuss the distribution for queue length L at the arrival epochs. From Theorem 2, we easily obtain the distribution for L

$$p\{L=0\} = \pi_{00} = (1-\xi)\sigma,$$
$$p\{L=k\} = \pi_{k0} + \pi_{k1} = (1-\xi)\sigma\left(\gamma^k + \alpha(\gamma-\xi)\sum_{j=0}^{k-1}\xi^j\gamma^{k-1-j}\right), \quad k \geqslant 1.$$

Theorem 4. *If $\rho < 1$ and $\mu > \eta$, the stationary queue length L can be decomposed into the sum of two independent random variables: $L = L_0 + L_d$, where L_0 is the stationary queue length of a classical GI/Geo/1 queue without vacation, follows a geometric distribution with parameter $1-\xi$. Additional queue length L_d has a modified geometric distribution*

$$P\{L_d = 0\} = \sigma,$$
$$P\{L_d = k\} = (1-\sigma)(1-\gamma)\gamma^{k-1}, \quad k \geqslant 1, \tag{14}$$

where $\gamma = A((1-\theta)(1-\eta))$, and σ is defined as in Theorem 2.

Proof. From the probability expression of L above, the probability generating function of L is as follows:

$$\begin{aligned} L(z) &= \sum_{k=0}^{\infty} z^k\pi_{k0} + \sum_{k=1}^{\infty} z^k\pi_{k1} = (1-\xi)\sigma\left[\frac{1}{1-\gamma z} + \alpha(\gamma-\xi)\frac{1}{1-\xi z}\frac{z}{1-\gamma z}\right] \\ &= \frac{1-\xi}{1-\xi z}\sigma\left[\frac{1}{1-\gamma z}(1-\xi z) + \alpha(\gamma-\xi)\frac{z}{1-\gamma z}\right] \\ &= \frac{1-\xi}{1-\xi z}\frac{1}{\delta}\left[1-\gamma+(\alpha+1)(\gamma-\xi)\frac{(1-\gamma)z}{1-\gamma z}\right] = \frac{1-\xi}{1-\xi z}\left[\sigma+(1-\sigma)\frac{(1-\gamma)z}{1-\gamma z}\right] \\ &= L_0(z)L_d(z). \end{aligned} \tag{15}$$

where $\delta = 1-\xi+\alpha(\gamma-\xi)$. In the equation above, we easily verify

$$1-\sigma = \frac{(\alpha+1)(\gamma-\xi)}{1-\xi+\alpha(\gamma-\xi)} = \frac{\mu-\eta}{\theta(1-\eta)+\eta\gamma}\frac{\alpha(\gamma-\xi)(1-\gamma)}{1-\xi+\alpha(\gamma-\xi)}. \qquad \square$$

Eq. (14) indicates that the additional delay L_d can be written as the mixture of two random variables: $L_d = \sigma X_0 + (1-\sigma)X_1$, where $X_0 \equiv 0$, X_1 follows a geometric distribution with parameter $1-\gamma$. Thus, we easily obtain

$$E(L) = \frac{\xi}{1-\xi} + \frac{\mu-\eta}{\theta(1-\eta)+\eta\gamma}\frac{\alpha(\gamma-\xi)}{1-\xi+\alpha(\gamma-\xi)}.$$

For the waiting time, we can easily verify that there is no complete stochastic decomposition property, but we can obtain the conditional stochastic decomposition structure when the server is in the busy period.

Denoting W_1 the conditional waiting time when the server is in the busy period, i.e., $J = 1$, we obtain

Theorem 5. *W_1 can be decomposed into the sum of two independent random variables: $W_1 = W_0 + W_d$, where W_0 is the waiting time of a classical GI/Geo/1 queue without vacation, follows a modified geometric distribution with parameter ξ. Additional queue length L_d has a geometric distribution with parameter $\mu(1-\gamma)$.*

Proof. From the definition of W_1, its PGF is as follows:

$$W_1(z) = \frac{1}{p\{J=1\}}\sum_{k=1}^{+\infty}\pi_{k1}\left\{\mu\left(\frac{\mu z}{1-(1-\mu)z}\right)^{k-1} + (1-\mu)\left(\frac{\mu z}{1-(1-\mu)z}\right)^{k}\right\}$$
$$= \frac{(1-\xi)(1-(1-\mu)z)}{1-(1-\mu(1-\xi))z}\,\frac{\mu(1-\gamma)}{1-(1-\mu(1-\gamma))z}.$$

From Eq. (10), we easily obtain the results and do not explain in detail. □

6. Conclusions

In this paper, we consider a discrete-time GI/Geo/1 queue with working vacations and vacation interruption and obtain the distributions for queue length and waiting time. With these indices of the system, we can model some practical problems in the communication networks and computer and evaluate the performance of those systems. Meanwhile, we also present the stochastic decomposition results for those indices and establish the theoretical framework for the GI/Geo/1 queue with working vacations and vacation interruption.

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References

[1] B.T. Doshi, Single server queues with vacations, in: H. Takagi (Ed.), Stochastic Analysis of the Computer and Communication Systems, North-Holland Elsevier, Amsterdam, 1990, pp. 217–264.
[2] B.T. Doshi, Queueing systems with vacations – a survey, Queueing Syst. 1 (1986) 29–66.
[3] H. Takagi, Queueing Analysis: A Foundation of Performance Evaluation, Vacation and Priority Systems, Part 1, vol. 1, North-Holland Elsevier, Amsterdam, 1991.
[4] N. Tian, Z.G. Zhang, Vacation Queueing Models: Theory and Applications, Springer-Verlag, New York, 2006.
[5] T. Meisling, Discrete time queueing theory, Oper. Res. 6 (1958) 96–105.
[6] J.J. Hunter, Mathematical techniques of applied probability, Discrete Time Models: Techniques and Applications, vol. 2, Academic Press, New York, 1983.
[7] H. Takagi, Queueing Analysis, Discrete-Time Systems, vol. 3, North-Holland Elsevier, Amsterdam, 1993.
[8] N. Tian, Z.G. Zhang, The discrete time GI/Geo/1 queue with multiple vacations, Queueing Syst. 40 (2002) 283–294.
[9] Z.G. Zhang, N. Tian, Geo/G/1 queue with multiple adaptive vacations, Queueing Syst. 38 (2001) 419–429.
[10] A.S. Alfa, Vacation models in discrete time, Queueing Syst. 44 (2003) 5–30.
[11] N. Tian, D. Zhang, C. Cao, The GI/M/1 queue with exponential vacations, Queueing Syst. 5 (1989) 331–344.
[12] U. Chatterjee, S. Mukherjee, GI/M/1 queue with server vacations, J. Operat. Res. Soc. 41 (1990) 83–87.
[13] N. Tian, Z.G. Zhang, A note on GI/M/1 queues with phase-type setup times or server vacations, INFOR 41 (2003) 341–351.
[14] L.D. Servi, S.G. Finn, M/M/1 queue with working vacations (M/M/1/WV), Perform. Evaluation 50 (2002) 41–52.
[15] J.D. Kim, D.W. Choi, K.C. Chae, Analysis of queue-length distribution of the M/G/1 queue with working vacations, in: Hawaii International Conference on Statistics and Related Fields, June 5–8, 2003.
[16] D. Wu, H. Takagi, M/G/1 queue with multiple working vacations, in: Proceedings of the Queueing Symposium, Stochastic Models and the Applications, Kakegawa, 2003, pp. 51–60.
[17] Y. Baba, Analysis of a GI/M/1 queue with multiple working vacations, Operat. Res. Lett. 33 (2005) 201–209.
[18] J. Li, W. Liu, N. Tian, The discrete-time GI/Geo/1 queue with multiple working vacations, Queueing Sys., submitted for publication.
[19] M. Neuts, Matrix-Geometric Solutions in Stochastic models, Johns Hopkins University Press, Baltimore, 1981.