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Discrete-time GI/Geo/1 queue with multiple working vacations

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Abstract Consider the discrete time GI/Geo/1 queue with working vacations under EAS and LAS schemes. The server takes the original work at the lower rate rather than completely stopping during the vacation period. Using the matrix-geometric solution method, we obtain the steady-state distribution of the number of customers in the system and present the stochastic decomposition property of the queue length. Furthermore, we find and verify the closed property of conditional probability for negative binomial distributions. Using such property, we obtain the specific expression for the steady-state distribution of the waiting time and explain its two conditional stochastic decomposition structures. Finally, two special models are presented.

Keywords Discrete-time · Working vacations · Matrix-geometric approach · Closed property of conditional probability · Stochastic decomposition

Mathematics Subject Classification (2000) 60K25

1 Introduction

During the last two decades, the queueing systems with server vacations have been well investigated. In the models with various vacation policies, the server completely stops service in the vacation period, but he can take the assistant work. Research work of the vacation queues has been extensively used in computer networks, communication systems and production management et al. Details can be seen in surveys of Doshi [5, 6] and the monographs of Takagi [15] and Tian and Zhang [19].

Parallel to continuous time queues, discrete-time queues where the inter-arrival time and service time are positive integer random variables have been analyzed and are more suitable to model and analyze the digital communication system. Meisling [12] firstly presented the discrete time queue. Hunter [7] collected the early results of the discrete time queues. Takagi [16] gave the analysis of various kinds of Geo/G/1 queues. Tian and Zhang [17], Zhang and Tian [22] investigated a GI/Geo/1 with multiple vacations and a Geo/G/1 queue with multiple adaptive vacations, respectively. Alfa [1] systematically analyzed a series of models with non-exhaustive service in which both the vacation time and service time follow phase type distributions.

In all queueing systems with vacations those authors considered, the server stops the service completely during the vacation period. Recently, Servi and Finn [14] first studied an M/M/1 queue with the working vacation policy: the server can work at the lower speed during the vacation period rather than stopping completely. Their work is motivated and illustrated by the analysis of a WDM optical access network using multiple wavelengths which can be reconfigured. Subsequently, Kim, Choi and Chae [9], Wu and Takagi [21] generalized results in [14] to an M/G/1 queue with working vacations. Baba [2] extended this study to a GI/M/1 queue with working vacations by the matrix-analysis method. Banik et al. [3] analyzed the GI/M/1/N queue with working vacations. Recently, parallel to the results of the continuous time M/M/1 queue with working vacations (WV), Li and Tian [11] researched the Geo/Geo/1

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queue with single working vacation, and gave various kinds of stationary distributions and presented stochastic decomposition structures of the stationary indices.

Compared to the M/G/1 queues with vacations, the studies on GI/M/1 type queues with vacations are relatively small. Initially, Tian et al. [20] used the matrix geometric solution method to analyze. They obtained expressions for the rate matrix and proved stochastic decomposition properties for the queue length and waiting time in a GI/M/1 vacation model with multiple exponential vacations. Independently, Chatterjee and Mukherjee [4] also researched the GI/M/1 with server vacations. Subsequently, Tian and Zhang [17, 18] discussed the discrete time GI/Geo/1 queue with server vacations and the GI/M/1 queue with PH (Phase-type) vacations or setup times, respectively. Some authors considered the GI/M/1 queues with finite buffer and the other policies and details can be seen in [8, 10]. Recently, as we state above, Baba [2] and Banik et al. [3] considered the GI/M/1 queue and GI/M/1/N queue with working vacations, respectively.

In this paper, we will extend the GI/M/1 queue with working vacations in [2] to the discrete-time GI/Geo/1 queue with working vacations (WV). In discrete-time epochs, the customer is served at the lower rate during the vacation period. Such model with working vacations has some certain implications in practice: (1) Servi and Finn [14] illustrated that the model with working vacations can be used to analyze the performance of a WDM optical access network. But, in the digital communication systems, the messages are always divided into units and their arrivals and departures occur at certain fixed time epochs. Therefore, discrete-time models are more suitable to the practical working situation in Internet Protocol (IP) access networks. Meanwhile, in the cyclic service queue model which is always used to reconfigure the communication network, we also can apply the working vacation policy to model; (2) The essence of the working vacation policy is that, when the number of customers is less relatively, a "lower speed period" is established to economize the operational cost in the system. The analysis of the model with working vacations can provide the theory and analysis method to design the optimal lower speed period.

It is well known that all discrete-time queues have two variations due to the specific nature of the arrivals and departures. If at a certain time point, the departure occurs before the arrival, i.e.,the service completion occurs at the end of the time interval, with any new arrival joining the system at the beginning of the time interval, it is called the early arrival system (EAS). Alternatively, an arrival is before a departure and the arriving customer is blocked from entering an empty service facility until the servicing interval terminates, in other words, the service of an arrival starts in the next interval rather than in the arrival interval immediately, then it is called the late arrival system with delayed access (LAS-DA). And, in the late arrival system with immediate access (LAS-IA), the arriving item can enter the service facility, if it is free, for an immediate initial unit of service. This is also equivalent to EAS. The specific discussion for EAS, LAS-IA and LAS-DA also can be seen in Hunter [7] and we also will make more intuitionistic assumption later.

In this paper, using the matrix-geometric approach, we discuss the GI/Geo/1 queue with working vacations and obtain the uniform results under EAS (LAS-IA) and LAS-DA schemes. In Sect. 2, we provide the model formulation and its structural matrix. In Sect. 3, we give the steady-state distribution of the number of customers and present the stochastic decomposition property for the queue length. In Sect. 4, we verify the closed property of conditional probability for negative binomial distributions, and obtain the distribution of the waiting time and two conditional stochastic decomposition structures. Finally, two special models are considered in Sect. 5.

2 Model formulation and structural matrix

We consider a single-server discrete-time queue with a general arrival process and a geometric service. It is assumed that time is slotted in intervals of equal length with the length of the slot being unity. First, we define the early arrival system (EAS) and assume that potential customer arrivals occur in (m, m^+) and potential departures take place in (m^-, m). Alternatively, in the late arrival system (LAS), arrivals occur in (m^-, m) and potential departures take place in (m, m^+). Certainly, as we state above, there are two types of the late arrival systems, LAS-DA and LAS-IA. And, LAS-IA is equivalent to EAS. Thus, we only consider the systems under the EAS and LAS-DA schemes. To distinguish two systems, we denote

$$\delta = \begin{cases} 0, & \text{the early arrival system (EAS),} \\ 1, & \text{the late arrival system} \\ & \text{with delayed access (LAS-DA).} \end{cases}$$

For such two systems, inter-arrival times $\{T_n, n \geq 1\}$ are independent and identically distributed (i.i.d.) sequences.

$$P\{T_n = j\} = \lambda_j, \quad j \geq 1;$$

$$E(T_n) = 1/\lambda; \qquad A^*(z) = \sum_{j=1}^{\infty} \lambda_j z^j.$$

The service times $\{S_n, n \geq 1\}$ and $\{S_n^*, n \geq 1\}$ in a regular busy period and a working vacation period follow geometric distributions with parameters μ, η $(0 \leq \eta \leq \mu < 1)$, respectively.

$$P\{S_n = j\} = \mu(1-\mu)^{j-1}, \quad j \geq 1;$$
$$P\{S_n^* = j\} = \eta(1-\eta)^{j-1}, \quad j \geq 1.$$

For such a model, a server begins a working vacation at the instant when the queue becomes empty. Suppose that the beginning and ending of vacations also occur at the instant $t = m^+$ (EAS) or $t = m^-$ (LAS-DA). Vacation time V follows geometric distributions:

$$P\{V = j\} = \theta(1-\theta)^{j-1}, \quad j \ge 1;$$
$$E(V) = \theta^{-1}, \quad 0 < \theta < 1.$$

During a working vacation arriving customers are served according to arrival order by the rate η; when a working vacation ends, if there are customers in the queue, the server switches the service rate from η to μ, and a regular busy period starts. Otherwise, the server keeps on another working vacation.

We assume that inter-arrival times, service times, and working vacation times are mutually independent. The service discipline is first in first out (FIFO).

Let L_n be the number of customers in the system at the nth arrival instant $t = m^+$ for EAS or $t = m^-$ for LAS-DA. Define

$$J_n = \begin{cases} 0, & \text{the } n\text{th arrival occurs during} \\ & \text{a working vacation period,} \\ 1, & \text{the } n\text{th arrival occurs during a service period.} \end{cases}$$

Then the process $\{(L_n, J_n), n \ge 1\}$ is an embedded Markov chain with the state space

$$\Omega = \{(0,0)\} \cup \{(k, j), k \ge 1, j = 0, 1\}.$$

In order to express the transition probability matrix of (L_n, J_n), let

$$P_{(i,j),(k,l)} = P(L_{n+1} = k, J_{n+1} = l \mid L_n = i, J_n = j).$$

Meanwhile, we introduce the expressions below

$$a_j = \sum_{r=j}^{\infty} \lambda_r \binom{r}{j} \mu^j (1-\mu)^{r-j};$$

$$b_j = \sum_{r=j}^{\infty} \lambda_r (1-\theta)^r \binom{r}{j} \eta^j (1-\eta)^{r-j};$$

$$c_j = \sum_{r=j}^{\infty} \lambda_r \sum_{i=1}^{r} \theta(1-\theta)^{i-1} \times \sum_{k=\Delta}^{\min(i,j)} \binom{i}{k} \eta^k (1-\eta)^{i-k} \times \binom{r-i}{j-k} \mu^{j-k} (1-\mu)^{r-i-j+k};$$

where we assume $\lambda_0 = 0$, $\Delta = \max(0, i + j - r)$, and $a_j, j \ge 0$ represent the probability that there are j service completions during an inter-arrival time in the busy period; $b_j, j \ge 0$ represent the probability that the working vacation time is longer than an inter-arrival time and there are j service completions during an inter-arrival time; $c_j, j \ge 0$ represent the probability that the working vacation time is not longer than an inter-arrival time and there are j service completions during an inter-arrival time.

Now, we consider the transition probabilities of (L_n, J_n). First, considering the case during a service period, the transition from $(i, 1)$ to $(j, 1)$ occurs if there are $i+1-j$ service completions during an inter-arrival time. Therefore, we have

$$P_{(i,1),(j,1)} = a_{i+1-j}, \quad i \ge 1, 1 \le j \le i+1.$$

Second, the transition from $(i, 0)$ to $(j, 0)$ occurs if the working vacation time is longer than an inter-arrival time and there are $i + 1 - j$ service completions during an inter-arrival time. Thus, we have

$$P_{(i,0),(j,0)} = b_{i+1-j}, \quad i \ge 1, 1 \le j \le i+1.$$

Third, the transition from $(i, 0)$ to $(j, 1)$ occurs if the working vacation time is not longer than an inter-arrival time and there are $i+1-j$ service completions during an inter-arrival time. Then, we have

$$P_{(i,0),(j,1)} = c_{i+1-j}, \quad i \ge 1, 1 \le j \le i+1.$$

From the above equations, we obtain

$$P_{(i,1),(0,0)} = 1 - \sum_{j=0}^{i} a_j, \quad i \ge 1;$$

$$P_{(i,0),(0,0)} = 1 - \sum_{j=0}^{i} (b_j + c_j), \quad i \ge 1.$$

Meanwhile, for $P_{(0,0),(1,0)}$ and $P_{(0,0),(1,1)}$, under the LAS-DA scheme, we have

$$P_{(0,0),(1,0)} = \sum_{r=1}^{\infty} \lambda_r (1-\theta)^r (1-\eta)^{r-1} = \frac{b_0}{1-\eta};$$

$$P_{(0,0),(1,1)} = \sum_{r=1}^{\infty} \lambda_r \sum_{i=1}^{r} \theta(1-\theta)^{i-1} (1-\eta)^i (1-\mu)^{r-i-1} = \frac{c_0}{1-\mu}.$$

Similarly, under the EAS scheme, $P_{(0,0),(1,0)} = b_0$, $P_{(0,0),(1,1)} = c_0$. Thus, note

$$P_{(0,0),(1,0)} = \frac{b_0}{1-\eta\delta}, \qquad P_{(0,0),(1,1)} = \frac{c_0}{1-\mu\delta}.$$

Using the lexicographical sequence for the states, the transition probability matrix of (L_n, J_n) can be written as the Block–Jacobi matrix

$$\widetilde{\mathrm{P}} = \begin{bmatrix} B_{00} & A_{01} & & & \\ B_1 & A_1 & A_0 & & \\ B_2 & A_2 & A_1 & A_0 & \\ B_3 & A_3 & A_2 & A_1 & A_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

where

$$B_{00} = 1 - \frac{b_0}{1-\eta\delta} - \frac{c_0}{1-\mu\delta};$$

$$A_{01} = \left(\frac{b_0}{1-\eta\delta}, \frac{c_0}{1-\mu\delta}\right);$$

$$A_k = \begin{bmatrix} b_k & c_k \\ 0 & a_k \end{bmatrix}, \quad k \ge 0;$$

$$B_k = \begin{bmatrix} 1 - \sum_{i=0}^{k}(c_i + b_i) \\ 1 - \sum_{i=0}^{k} a_i. \end{bmatrix}, \quad k \ge 1.$$

The matrix $\widetilde{\mathrm{P}}$ in (1) is a GI/M/1 type matrix (see [13]). To analyze the GI/M/1 type system, the minimal nonnegative solution R of the matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$, called the rate matrix is important. To solve R, we need the following lemmas.

Lemma 1 *If $\rho = \lambda/\mu < 1$ and $0 < \theta \le 1, 0 \le \eta \le \mu < 1$, then the equations $z = A^*(1-\mu(1-z))$ and $z = A^*((1-\theta)(1-\eta(1-z)))$ have the respective unique roots in the range $0 < z < 1$.*

Proof First, we consider the equation $z = A^*(1-\mu(1-z))$. Let $\psi(z) = A^*(1-\mu(1-z))$ and evidently, $0 < \psi(0) = A^*(1-\mu) < \psi(1) = 1$. And, for $0 < z < 1$,

$$\psi'(z) = \mu A^{*\prime}(1-\mu(1-z)) > 0;$$

$$\psi''(z) = \mu^2 A^{*(2)}(1-\mu(1-z)) > 0.$$

Meanwhile, from $\rho = \lambda/\mu < 1$, $\psi'(1) = \rho^{-1} > 1$. Thus, the equation $z = \psi(z)$ has a unique root in the range $0 < z < 1$.

Similarly, we set $\varphi(z) = A^*((1-\theta)(1-\eta(1-z)))$. Then we have

$$0 < \varphi(0) = A^*((1-\theta)(1-\eta)) < \varphi(1) = A^*(1-\theta) < 1.$$

And, for $0 < z < 1$,

$$\varphi'(z) = \eta(1-\theta)A^{*\prime}((1-\theta)(1-\eta(1-z))) > 0;$$

$$\varphi''(z) = \eta^2(1-\theta)^2 A^{*(2)}((1-\theta)(1-\eta(1-z))) > 0.$$

It is shown that the equation $z = \varphi(z)$ has a unique root in the range $0 < z < 1$. □

Lemma 2 *If $\rho = \lambda/\mu < 1$ and $0 < \theta \le 1, 0 \le \eta \le \mu < 1$, then*

$$\beta(\gamma - \xi) = \frac{\theta[1-\eta(1-\gamma)]}{(1-\theta)[1-\eta(1-\gamma)] - [1-\mu(1-\gamma)]}(\gamma - \xi) > 0,$$

where ξ, γ are the unique roots in the range $0 < z < 1$ of the equations $z = A^*(1-\mu(1-z))$ and $z = A^*((1-\theta)(1-\eta(1-z)))$, respectively.

Proof From Lemma 1, if $\rho < 1$, the equation $z = A^*(1-\mu(1-z))$ has a unique root $z = \xi, 0 < \xi < 1$, and there are two cases. Case 1: for $0 < z < \xi$, we have $z < A^*(1-\mu(1-z))$; case 2: for $\xi < z < 1$, we have $z > A^*(1-\mu(1-z))$. Assuming that $0 < (1-\theta)[1-\eta(1-\gamma)] < 1-\mu(1-\xi)$ and taking the probability generating function of this inequality, we obtain

$$A^*((1-\theta)(1-\eta(1-\gamma))) < A^*(1-\mu(1-\xi)) = \xi.$$

Thus we have $0 < \gamma < \xi$. Hence, we must have $(1-\theta) \times (1-\eta(1-\gamma)) < 1-\mu(1-\gamma)$. Otherwise, $(1-\theta) \times (1-\eta(1-\gamma)) \ge 1-\mu(1-\gamma)$ results in

$$\gamma = A^*((1-\theta)(1-\eta(1-\gamma))) \ge A^*(1-\mu(1-\gamma)).$$

This inequality is a contradiction to case 1. So $\gamma - \xi$ and $(1-\theta)(1-\eta(1-\gamma)) - [1-\mu(1-\gamma)]$ are negative and $\beta(\gamma-\xi) > 0$. Similarly, we can prove this inequality for $1-\mu(1-\xi) < (1-\theta)[1-\eta(1-\gamma)] < 1$. □

Then we obtain an important lemma.

Lemma 3 *If $\rho = \lambda/\mu < 1$ and $0 < \theta \le 1, 0 \le \eta \le \mu < 1$, then the matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$ has the minimal nonnegative solution*

$$R = \begin{bmatrix} \gamma & \beta(\gamma - \xi) \\ 0 & \xi \end{bmatrix},$$

where ξ, γ are the unique roots in the range $0 < z < 1$ of the equations $z = A^*(1-\mu(1-z))$ and $z = A^*((1-\theta)(1-\eta(1-z)))$, respectively. β is defined as in Lemma 2.

Proof Because all A_k are upper triangular, we can assume that R has the same structure as

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}.$$

Then, for $k \ge 1$, we have

$$R^k = \begin{bmatrix} r_{11}^k & r_{12} \sum_{j=0}^{k-1} r_{11}^j r_{22}^{k-1-j} \\ 0 & r_{22}^k \end{bmatrix}.$$

Substituting R^k into $R=\sum_{k=0}^{\infty} R^k A_k$, we obtain

$$\begin{cases} r_{11}=\sum_{k=0}^{\infty} b_k r_{11}^k, \\ r_{12}=\sum_{k=0}^{\infty} r_{11}^k c_k + r_{12}\sum_{k=0}^{\infty} a_k \sum_{j=0}^{k-1} r_{11}^j r_{22}^{k-1-j}, \\ r_{22}=\sum_{k=0}^{\infty} a_k r_{22}^k = A^*(1-\mu(1-r_{22})). \end{cases} \tag{2}$$

Evidently, from Lemma 1, if $\rho=\lambda/\mu<1$, the third equation has the unique root $r_{22}=\xi$ in the range $0<r_{22}<1$. For the first equation, we compute

$$\begin{aligned} r_{11} &= \sum_{k=0}^{\infty}\sum_{r=k}^{\infty}\lambda_r \sum_{i=r+1}^{\infty}\theta(1-\theta)^{i-1}\binom{r}{k}\eta^k(1-\eta)^{r-k} r_{11}^k \\ &= \sum_{r=1}^{\infty}\lambda_r \sum_{i=r+1}^{\infty}\theta(1-\theta)^{i-1}\sum_{k=0}^{r}\binom{r}{k}\eta^k(1-\eta)^{r-k} r_{11}^k \\ &= \sum_{r=1}^{\infty}\lambda_r(1-\theta)^r[1-\eta(1-r_{11})]^r \\ &= A^*((1-\theta)(1-\eta(1-r_{11}))). \end{aligned}$$

From Lemma 1, the equation $z=A^*((1-\theta)(1-\eta(1-z)))$ has the unique root $r_{11}=\gamma$ in the range of $0<z<1$. Substituting $r_{22}=\xi$ and $r_{11}=\gamma$ into the second equation, we have

$$r_{12}\left(1-\sum_{k=0}^{\infty} a_k \sum_{j=0}^{k-1}\gamma^j \xi^{k-1-j}\right)=\sum_{k=0}^{\infty}\gamma^k c_k. \tag{3}$$

And, we easily compute

$$\begin{aligned} &\sum_{k=0}^{\infty}\gamma^k c_k \\ &= \sum_{r=0}^{\infty}\lambda_r \sum_{i=1}^{r}\theta(1-\theta)^{i-1} \\ &\quad \times \sum_{k=0}^{r}\sum_{l=0}^{i}\binom{i}{l}\eta^l(1-\eta)^{i-l} \\ &\quad \times \binom{r-i}{k-l}\mu^{k-l}(1-\mu)^{r-i-k+l}\gamma^k \\ &= \sum_{r=0}^{\infty}\lambda_r[1-\mu(1-\gamma)]^r \\ &\quad \times \sum_{i=1}^{r}\theta(1-\theta)^{i-1}[1-\mu(1-\gamma)]^{-i}[1-\eta(1-\gamma)]^i \\ &= \frac{\theta(1-\eta(1-\gamma))}{(1-\theta)(1-\eta(1-\gamma))-(1-\mu(1-\gamma))} \\ &\quad \times (\gamma - A^*(1-\mu(1-\gamma))) \\ &= \beta(\gamma - A^*(1-\mu(1-\gamma))), \end{aligned}$$

and

$$\begin{aligned} 1-\sum_{k=0}^{\infty} a_k \sum_{j=0}^{k-1}\gamma^j\xi^{k-1-j} &= 1-\sum_{k=0}^{\infty} a_k \frac{\xi^k-\gamma^k}{\xi-\gamma} \\ &= \frac{\gamma - A^*(1-\mu(1-\gamma))}{\gamma-\xi}. \end{aligned}$$

Substituting the above equations into (3), we get r_{12} and finally obtain the expression for R. With R, we can obtain the steady-state condition and the distribution of the queue length as we explain in the next section. □

3 Steady-state distribution at arrival epochs

Then, we derive the steady-state distribution at arrival epochs by using Neuts' matrix-geometric approach. In order to derive the steady-state distribution, we need the theorem below.

Theorem 1 *The Markov chain $\widetilde{\mathrm{P}}$ is positive recurrent if and only if $\rho<1$ and $0<\theta\le 1, 0\le\eta\le\mu<1$.*

Proof Based on Theorem 1.5.1 of Neuts (1981) [13], the Markov chain $\widetilde{\mathrm{P}}$ is positive recurrent if and only if the spectral radius $\mathrm{SP}(R)$ of the rate matrix R is less than 1, and the matrix

$$B[R]=\begin{bmatrix} B_{00} & A_{01} \\ \sum_{k=1}^{\infty} R^{k-1}B_k & \sum_{k=1}^{\infty} R^{k-1}A_k \end{bmatrix}$$

has a positive left invariant vector. From Lemma 1, if $\rho<1$ and $0<\theta\le 1, 0\le\eta\le\mu<1$, evidently, $SP(R)=\max\{\gamma,\xi\}<1$. Substituting the expressions for R, A_k and B_k into $B[R]$, we obtain

$$B[R] = \begin{bmatrix} 1-\frac{b_0}{1-\eta\delta}-\frac{c_0}{1-\mu\delta} & \frac{b_0}{1-\eta\delta} & \frac{c_0}{1-\mu\delta} \\ \frac{b_0}{\gamma}+\frac{c_0}{\gamma}-\frac{a_0\beta(\gamma-\xi)}{\gamma\xi} & 1-\frac{b_0}{\gamma} & \frac{a_0\beta(\gamma-\xi)}{\gamma\xi}-\frac{c_0}{\gamma} \\ \frac{a_0}{\xi} & 0 & 1-\frac{a_0}{\xi} \end{bmatrix}.$$

It can be easily verified that $B[R]$ has the left invariant vector

$$x_0\left(1, \frac{\gamma}{1-\eta\delta}, \frac{\beta(\gamma-\xi)+w_0\xi}{1-\eta\delta}\right), \tag{4}$$

where x_0 is any positive real number and

$$w_0=\frac{c_0}{a_0}\frac{(\mu-\eta)\delta}{1-\mu\delta}.$$

Certainly, under the EAS scheme, $w_0=0$. Thus, if $\rho<1$ and $0<\theta\le 1, 0\le\eta\le\mu<1$, the Markov chain $\widetilde{\mathrm{P}}$ is positive recurrent. □

If $\rho < 1$, let (L, J) be the stationary limit of the Markov chain (L_n, J_n) under the EAS or LAS-DA scheme. Let

$$\pi_0 = \pi_{00}; \qquad \pi_k = (\pi_{k0}, \pi_{k1}), \quad k \geq 1;$$

$$\pi_{kj} = \mathrm{P}\{L = k, J = j\} = \lim_{n\to\infty} \mathrm{P}\{L_n = k, J_n = j\}, \quad (k, j) \in \Omega.$$

Theorem 2 *If $\rho < 1$, the stationary probability distribution of (L, J) is*

$$\begin{cases} \pi_{00} = \sigma(1-\xi)(1-\eta\delta), \\ \pi_{k0} = \sigma(1-\xi)\gamma^k, & k \geq 1, \\ \pi_{k1} = \sigma(1-\xi)[\beta(\gamma^k - \xi^k) + w_0\xi^k], & k \geq 1, \end{cases} \tag{5}$$

where

$$\sigma = \frac{1-\gamma}{(1-\xi)(1-\eta\delta(1-\gamma)) + \beta(\gamma-\xi) + w_0\xi(1-\gamma)}.$$

Proof With Theorem 1.5.1 of Neuts (see [13]), $(\pi_{00}, \pi_{10}, \pi_{11})$ is given by the positive left invariant vector (4) and satisfies the normalizing condition

$$\pi_{00} + (\pi_{10}, \pi_{11})(I - R)^{-1}e = 1.$$

And, substituting R into the above relation, we easily get

$$\pi_{00} = (1-\xi)(1-\eta\delta) \times \frac{1-\gamma}{(1-\xi)(1-\eta\delta(1-\gamma)) + \beta(\gamma-\xi) + w_0\xi(1-\gamma)} = \sigma(1-\xi)(1-\eta\delta).$$

Therefore, we obtain

$$(\pi_{10}, \pi_{11}) = \sigma(1-\xi)(\gamma, \beta(\gamma-\xi) + w_0\xi).$$

Using Theorem 1.5.1 of Neuts, we have

$$\pi_k = (\pi_{k0}, \pi_{k1}) = (\pi_{10}, \pi_{11})R^{k-1}, \quad k \geq 1. \tag{6}$$

Taking (π_{10}, π_{11}) and R^{k-1} into (6), we easily obtain the theorem. □

Thus, we easily obtain the state probabilities of a server in steady-state.

$$\mathrm{P}\{J = 0\} = \sum_{k=0}^{\infty} \pi_{k0} = \frac{(1-\xi)(1-\eta\delta(1-\gamma))}{(1-\xi)(1-\eta\delta(1-\gamma)) + \beta(\gamma-\xi) + w_0\xi(1-\gamma)},$$

$$\mathrm{P}\{J = 1\} = \sum_{k=1}^{\infty} \pi_{k1} = \frac{\beta(\gamma-\xi) + w_0\xi(1-\gamma)}{(1-\xi)(1-\eta\delta(1-\gamma)) + \beta(\gamma-\xi) + w_0\xi(1-\gamma)}.$$

Meanwhile, the stationary distribution of the number of customers in the system is given by

$$\mathrm{P}\{L = 0\} = \pi_{00} = \sigma(1-\xi)(1-\eta\delta),$$

$$\mathrm{P}\{L = k\} = \pi_{k0} + \pi_{k1} = \sigma(1-\xi)[\gamma^k + \beta(\gamma^k - \xi^k) + w_0\xi^k], \quad k \geq 1.$$

We can obtain the stochastic decomposition result for the queue length at arrival epochs.

Theorem 3 *If $\rho < 1$ and $\mu > \eta$, the stationary queue length L can be decomposed into the sum of two independent random variables: $L = L_0 + L_d$, where L_0 is the stationary queue length of a classical GI/Geo/1 queue without vacations under the EAS scheme, and follows a geometric distribution with the parameter $1-\xi$; additional queue length L_d has the probability generating function (PGF)*

$$L_d(z) = \sigma_1 + \sigma_2 z + (1 - \sigma_1 - \sigma_2)\frac{(1-\gamma)z}{1-\gamma z},$$

where $\sigma_1 = \sigma(1-\eta\delta)$, $\sigma_2 = \sigma(w_0 + \eta\delta)\xi$.

Proof From the probability expression for L above, the probability generating function of L is as follows

$$\begin{aligned} L(z) &= \pi_{00} + \sum_{k=1}^{\infty} z^k \pi_{k0} + \sum_{k=1}^{\infty} z^k \pi_{k1} \\ &= \sigma(1-\xi)\Bigg[(1-\eta\delta) + \frac{\gamma z}{1-\gamma z} + \beta(\gamma-\xi)\frac{1}{1-\xi z}\frac{z}{1-\gamma z} + w_0\frac{\xi z}{1-\xi z}\Bigg] \\ &= \frac{1-\xi}{1-\xi z}\sigma\Bigg[(1-\eta\delta)(1-\xi z) + \frac{(1-\xi z)\gamma z}{1-\gamma z} + \beta(\gamma-\xi)\frac{z}{1-\gamma z} + w_0\xi z\Bigg] \\ &= \frac{1-\xi}{1-\xi z}\sigma\Bigg[(1-\eta\delta) + (w_0+\eta)\xi z + (\beta+1)(\gamma-\xi)\frac{z}{1-\gamma z}\Bigg] \\ &= \frac{1-\xi}{1-\xi z}\Bigg[\sigma_1 + \sigma_2 z + \sigma_3\frac{(1-\gamma)z}{1-\gamma z}\Bigg] = L_0(z)L_d(z). \end{aligned} \tag{7}$$

Evidently, we can verify easily

$$\sigma_3 = \frac{(\beta+1)(\gamma-\xi)}{(1-\xi)(1-\eta\delta(1-\gamma)) + \beta(\gamma-\xi) + w_0\xi(1-\gamma)} = 1 - \sigma_1 - \sigma_2.$$

□

Remark 1 We should mention that $L_0(z) = (1-\xi)/(1-\xi z)$ in (7) is the queue length distribution of the classical GI/Geo/1 queue without vacations under the EAS (LAS-IA) scheme. Under the LAS-DA scheme,

$$L_{\mathrm{LAS}}(z) = L_0(z)\frac{1-\mu+\mu\xi z}{1-\mu+\mu\xi} = L_0(z)\left[\frac{1-\mu}{1-\mu+\mu\xi} + \frac{\mu\xi}{1-\mu+\mu\xi}z\right],$$

i.e., L_{LAS} can be decomposed into the sum of L_0 and a random variable with the two-point distribution. In our models with working vacations, from (7), we also can demonstrate that the stationary queue length L can be decomposed into the sum of L_0 and an additional queue length L_d. The additional queue length L_d has specific probability significance. With the probability σ_1, it equals 0; with the probability σ_2, it equals 1; and it follows the geometric distribution with the parameter γ with the probability $1-\sigma_1-\sigma_2$.

4 Waiting time distribution

4.1 Conditional probability for negative binomial distributions

To analyze the steady-state waiting time, we firstly demonstrate the closed property of conditional probability for negative binomial distributions.

Assume that X follows the negative binomial distribution with parameters r and p, and V follows the geometric distribution with the parameter θ, i.e.,

$$\mathrm{P}\{X=m\} = \binom{m-1}{r-1} p^r(1-p)^{m-r}, \quad m \ge r;$$
$$\mathrm{P}\{V=k\} = \theta(1-\theta)^k, \quad k \ge 0.$$

We have two lemmas to present the closed property of conditional probability for negative binomial distributions.

Lemma 4.1 *If X and V are mutually independent, under the condition $X \le V$, X follows the negative binomial distribution with parameters r and $\theta + p(1-\theta)$.*

Proof First, we compute the conditional probability

$$\begin{aligned}
&\mathrm{P}\{X \le V\} \\
&\quad = \sum_{m=r}^{\infty} \mathrm{P}\{X=m\}\mathrm{P}\{V \ge m\} \\
&\quad = \sum_{m=r}^{\infty} \binom{m-1}{r-1} \\
&\qquad \times p^r(1-p)^{m-r}(1-\theta)^m \quad (k=m-r+1) \\
&\quad = [p(1-\theta)]^r \sum_{k=1}^{\infty} \binom{k-1+r-1}{r-1}[(1-p)(1-\theta)]^{k-1} \\
&\quad = \left[\frac{p(1-\theta)}{\theta+p(1-\theta)}\right]^r.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{P}\{X=k|X \le V\} \\
&\quad = \frac{\mathrm{P}\{X=k, X \le V\}}{\mathrm{P}\{X \le V\}} \\
&\quad = \left[\frac{\theta+p(1-\theta)}{p(1-\theta)}\right]^r \binom{k-1}{r-1} p^r(1-p)^{k-r}(1-\theta)^k \\
&\quad = \binom{k-1}{r-1}[\theta+p(1-\theta)]^r\{1-[\theta+p(1-\theta)]\}^{k-r}.
\end{aligned}$$

Thus, we can find that under the condition $X \le V$, X follows negative binomial distribution with parameters r and $\theta + p(1-\theta)$. □

So, we obtain the PGF of X under the condition $X \le V$,

$$\Psi_r(z) = \left(\frac{z[\theta+p(1-\theta)]}{1-z(1-[\theta+p(1-\theta)])}\right)^r.$$

We present that, under the condition $X \le V$, X also follows the negative binomial distribution and we call it the closed property of the conditional probability of negative binomial distributions.

Meanwhile, assuming Y follows the geometric distribution with the parameter p, i.e.,

$$\mathrm{P}\{Y=j\} = p(1-p)^{j-1}, \quad j \ge 1,$$

we can verify the closed property of another conditional probability.

Lemma 4.2 *If X and V are mutually independent, under the condition $X \le V < X+Y$, V can be composed into the sum of two independent random variables X^*, Y^*: X^* follows the geometric distribution with the parameter $\theta + p(1-\theta)$; Y^* follows the negative binomial distribution with parameters r and $\theta + p(1-\theta)$.*

Proof Similarly, we have

$$\begin{aligned}
&\mathrm{P}\{X \le V < X+Y\} \\
&\quad = \sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r(1-p)^{k-r} \sum_{j=k}^{\infty} \theta(1-\theta)^j(1-p)^{j-k} \\
&\quad = \theta \sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r(1-p)^{k-r}(1-\theta)^k \\
&\qquad \times \sum_{j=k}^{\infty}(1-\theta)^{j-k}(1-p)^{j-k} \\
&\quad = \frac{\theta}{\theta+p(1-\theta)}\left[\frac{p(1-\theta)}{\theta+p(1-\theta)}\right]^r.
\end{aligned}$$

And,

$$\begin{aligned}
&\mathrm{P}\{V=k|X\le V<X+Y\}\\
&\quad=\mathrm{P}\{V=k, X\le V<X+Y\}/\mathrm{P}\{X\le V<X+Y\}\\
&\quad=[\theta+p(1-\theta)]^{r+1}\sum_{j=r}^{k}\binom{j-1}{r-1}\\
&\qquad\times(1-p)^{j-r}(1-\theta)^{k-r}(1-p)^{k-j}.
\end{aligned}$$

For the above equation, we obtain the generating function

$$\begin{aligned}
&\Phi_r(z)\\
&\quad=[\theta+p(1-\theta)]^{r+1}\sum_{k=r}^{\infty}z^k\sum_{j=r}^{k}\binom{j-1}{r-1}\\
&\qquad\times(1-p)^{j-r}(1-\theta)^{k-r}(1-p)^{k-j}\\
&\quad=[\theta+p(1-\theta)]^{r+1}\\
&\qquad\times\sum_{j=r}^{\infty}\binom{j-1}{r-1}(1-p)^{j-r}z^j(1-\theta)^{j-r}\\
&\qquad\times\sum_{k=j}^{\infty}(1-\theta)^{k-j}(1-p)^{k-j}z^{k-j}\\
&\quad=\frac{\theta+p(1-\theta)}{1-[1-(\theta+p(1-\theta))]z}\\
&\qquad\times\left[\frac{[\theta+p(1-\theta)]z}{1-[1-(\theta+p(1-\theta))]z}\right]^r.
\end{aligned}$$

Thus, we get the result and such two lemmas demonstrate the closed property of conditional probability for negative binomial distributions. □

4.2 Waiting time distribution

Let W and $W^*(z)$ be the steady-state waiting time and its PGF, respectively. With Lemmas 4.1 and 4.2, we can obtain the distribution of W. The service discipline is first in first out (FIFO). Meanwhile, denote H_1 be the probability that the server is in the service period when the new customer arrives, and H_0 be the probability that the server is in the vacation period and the new customer should wait. We can easily compute

$$\begin{aligned}
H_1&=\sum_{k=1}^{\infty}\pi_{k1}\\
&=\frac{\beta(\gamma-\xi)+w_0\xi(1-\gamma)}{(1-\xi)(1-\eta\delta(1-\gamma))+\beta(\gamma-\xi)+w_0\xi(1-\gamma)},\\
H_0&=\sum_{k=1}^{\infty}\pi_{k0}\\
&=\frac{(1-\xi)\gamma}{(1-\xi)(1-\eta\delta(1-\gamma))+\beta(\gamma-\xi)+w_0\xi(1-\gamma)}.
\end{aligned}$$

Theorem 4 *If $\rho<1$ and $\theta>0$, $\eta\le\mu$, the PGF of stationary waiting time W is*

$$\begin{aligned}
&W^*(z)\\
&\quad=1-H_0-H_1+H_0\frac{[1-(1-\theta)(\eta\gamma+1-\eta)]z}{1-(1-\theta)(\eta\gamma+1-\eta)z}\\
&\qquad\times\left\{1-q+q\frac{\mu(1-\gamma)}{1-[1-\mu(1-\gamma)]z}\right\}\\
&\qquad+H_1\frac{(1-\xi)[1-(1-\mu)z]}{1-[1-\mu(1-\xi)]z}\\
&\qquad\times\left\{p\frac{\mu(1-\gamma)z}{1-[1-\mu(1-\gamma)]z}+(1-p)\frac{\mu z}{1-(1-\mu)z}\right\},
\end{aligned}\tag{8}$$

where

$$\begin{aligned}
p&=\frac{\beta(\gamma-\xi)}{\beta(\gamma-\xi)+w_0\xi(1-\gamma)};\\
q&=\frac{\theta}{1-(1-\theta)(\eta\gamma+1-\eta)}=\frac{\theta}{\theta+\eta(1-\theta)(1-\gamma)}.
\end{aligned}$$

Proof Firstly, we easily obtain the probability that a new customer should not wait.

$$\mathrm{P}\{W=0\}=\pi_{00}=\sigma(1-\xi)(1-\eta\delta)=1-H_0-H_1.\tag{9}$$

When a new customer arrives at the instant $t=m^-$ (LAS) or $t=m^+$ (EAS), if there are k customers and the server is in the busy period, the waiting time equals k services by the rate μ. Then, we easily have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\pi_{k1}W_{k1}^*(z)\\
&\quad=\sigma(1-\xi)\sum_{k=1}^{\infty}[\beta(\gamma^k-\xi^k)+w_0\xi^k]\left(\frac{\mu z}{1-(1-\mu)z}\right)^k\\
&\quad=\sigma(1-\xi)\\
&\qquad\times\left[\beta(\gamma-\xi)\frac{1-(1-\mu)z}{1-[1-\mu(1-\xi)]z}\frac{\mu z}{1-[1-\mu(1-\gamma)]z}\right.\\
&\qquad\left.+w_0\xi\frac{\mu z}{1-[1-\mu(1-\xi)]z}\right]\\
&\quad=\frac{(1-\xi)[1-(1-\mu)z]}{1-[1-\mu(1-\xi)]z}\\
&\qquad\times\sigma\left[\beta(\gamma-\xi)\frac{\mu z}{1-[1-\mu(1-\gamma])z}\right.\\
&\qquad\left.+w_0\xi\frac{\mu z}{1-(1-\mu)z}\right]\\
&\quad=H_1\frac{(1-\xi)[1-(1-\mu)z]}{1-[1-\mu(1-\xi)]z}
\end{aligned}$$

$$\times \left[p \frac{\mu(1-\gamma)z}{1-[1-\mu(1-\gamma)]z} + (1-p)\frac{\mu z}{1-(1-\mu)z} \right]. \tag{10}$$

Denote $S_v^{(j)}$ the sum of j service times S_v with the rate η, i.e., j-dimensional convolution of S_v, and evidently it follows negative binomial distribution with parameters j and η. If there are k customers and the server is in the vacation period when the new customer arrives, there are two cases. Case 1: if there are j customer service completions when the vacation ends, i.e., $S_v^{(j)} \le V < S_v^{(j+1)}, 0 \le j \le k-1$, the waiting time is the sum of the vacation time under the above condition and $k-j$ service times by the rate μ; case 2: if at least k customers are served when the vacation ends, i.e., $V \ge S_v^k$, the waiting time is k service times with the rate η under the above condition. From Lemmas 4.1 and 4.2, the PGF of the waiting time is

$$\begin{aligned}
&W_{k0}^*(z)\\
&= \mathrm{P}\{V \ge S_v^k\}\Psi_k(z)\\
&\quad + \sum_{j=0}^{k-1} \mathrm{P}\{S_v^{(j)} \le V < S_v^{(j+1)}\}\Phi_j(z)\left[\frac{\mu z}{1-(1-\mu)z}\right]^{k-j}\\
&= \left[\frac{\eta(1-\theta)z}{1-(1-\theta)(1-\eta)z}\right]^k\\
&\quad + \sum_{j=0}^{k-1} \frac{\theta}{1-(1-\theta)(1-\eta)z}\left[\frac{\eta(1-\theta)z}{1-(1-\theta)(1-\eta)z}\right]^j\\
&\quad \times \left[\frac{\mu z}{1-(1-\mu)z}\right]^{k-j}.
\end{aligned}$$

Thus, we easily compute

$$\begin{aligned}
&\sum_{k=1}^{\infty} \pi_{k0} W_{k0}^*(z)\\
&= \sigma(1-\xi)\sum_{k=1}^{\infty}\gamma^k \left\{ \left[\frac{\eta(1-\theta)z}{1-(1-\theta)(1-\eta)z}\right]^k \right.\\
&\quad + \sum_{j=0}^{k-1} \frac{\theta}{1-(1-\theta)(1-\eta)z}\left[\frac{\eta(1-\theta)z}{1-(1-\theta)(1-\eta)z}\right]^j\\
&\quad \left. \times \left[\frac{\mu z}{1-(1-\mu)z}\right]^{k-j} \right\}\\
&= \sigma(1-\xi)\gamma \left\{ \frac{\eta(1-\theta)z}{1-(1-\theta)(\eta\gamma+1-\eta)z} \right.\\
&\quad \left. + \frac{\theta}{1-(1-\theta)(\eta\gamma+1-\eta)z}\frac{\mu z}{1-[1-\mu(1-\gamma)]z} \right\}\\
&= H_0 \frac{[1-(1-\theta)(\eta\gamma+1-\eta)]z}{1-(1-\theta)(\eta\gamma+1-\eta)z}\\
&\quad \times \left\{ 1-q+q\frac{\mu(1-\gamma)}{1-[1-\mu(1-\gamma)]z} \right\}.
\end{aligned} \tag{11}$$

And, we easily have

$$W^*(z) = \pi_{00} + \sum_{k=1}^{\infty}\pi_{k1}W_{k1}^*(z) + \sum_{k=1}^{\infty}\pi_{k0}W_{k0}^*(z).$$

From (9–11), we have the result in Theorem 4. □

With the structure in Theorem 4, we can easily get the expected waiting time

$$\begin{aligned}
E(W) &= H_1\left[\frac{\xi}{\mu(1-\xi)} + p\frac{1}{\mu(1-\gamma)} + (1-p)\frac{1}{\mu}\right]\\
&\quad + H_0\frac{1}{\theta+\eta(1-\theta)(1-\gamma)}\left[1+\theta\frac{1-\mu(1-\gamma)}{\mu(1-\gamma)}\right].
\end{aligned}$$

Remark 2 When we consider the waiting time, it is possible that a vacation ends at the arrival instant, so we assume that the vacation time can be 0. Meanwhile, for (8), the steady-state waiting time of an arbitrary customer has the special probability explanation. The waiting time equals 0 with the probability $1-H_0-H_1$; with the probability H_1, it equals the sum of one geometric random variable with the rate $\mu(1-\gamma)$ and one modified geometric random variable with the rate $\mu(1-\xi)$; with the probability H_0, it equals the sum of one geometric random variable with the rate $[1-(1-\theta)(\eta\gamma+1-\eta)]$ and one random variable which is the mixture of two geometric random variables with parameters $\mu(1-\gamma)$ and μ, respectively.

For the waiting time, we can easily verify that there is no complete stochastic decomposition property, but under some conditions, we can obtain the conditional stochastic decomposition structures.

First, denoting W_1 the conditional waiting time when the server is in the busy period, i.e., $J=1$, we obtain

Theorem 5.1 *W_1 can be decomposed into the sum of two independent random variables: $W_1 = W_0 + W_d$, where W_0 is the waiting time of a classical GI/Geo/1 queue without vacation, and follows a modified geometric distribution with parameter ξ; additional delay W_d has the generating function*

$$W_{1d}^*(z) = p\frac{\mu(1-\gamma)z}{1-[1-\mu(1-\gamma)]z} + (1-p)\frac{\mu z}{1-(1-\mu)z}.$$

Proof From the definition of W_1, its PGF is as follows

$$W_1^*(z) = \frac{1}{\mathrm{P}\{J=1\}}\sum_{k=1}^{\infty}\pi_{k1}\left(\frac{\mu z}{1-(1-\mu)z}\right)^k.$$

From (10), we easily obtain the results; we don't explain in detail. □

Similarly, for W_2 the conditional waiting time when the server is in the vacation period and the new customer should wait, we also have

Theorem 5.2 *W_2 can be decomposed into the sum of two independent random variables, and its PGF is*

$$W_2^*(z) = \frac{[1-(1-\theta)(\eta\gamma+1-\eta)]z}{1-(1-\theta)(\eta\gamma+1-\eta)z} \times \left\{1-q+q\frac{\mu(1-\gamma)}{1-[1-\mu(1-\gamma)]z}\right\}.$$

The proof is similar to that of Theorem 5.1 and we don't explain here.

5 Some special models

In this paper, we analyze a GI/Geo/1 queue with working vacations under EAS (LAS-IA) and LAS-DA schemes, and with a notation δ, obtain the uniform expressions for the distributions of the queue length and waiting time. We can derive some results of the special models.

Example 1 GI/Geo/1 queue with WV under the EAS scheme.

If $\delta = 0$, we have $w_0 = 0$ and can obtain the results under the EAS scheme. The Markov chain (L, J) has the distribution

$$\pi_{k0} = \sigma(1-\xi)\gamma^k, \quad k \ge 0;$$
$$\pi_{k1} = \sigma(1-\xi)\beta(\gamma^k - \xi^k), \quad k \ge 1.$$

And, the stationary queue length $L = L_0 + L_d$, where the generating function of the additional delayed L_d is

$$L_d(z) = \sigma + (1-\sigma)\frac{(1-\gamma)z}{1-\gamma z}, \quad \sigma = \frac{1-\gamma}{1-\xi+\beta(\gamma-\xi)}.$$

And, the expected queue length at arrival epochs is also obtained.

$$E(L) = \frac{\xi}{1-\xi} + \frac{(\beta+1)(\gamma-\xi)}{1-\xi+\beta(\gamma-\xi)}\frac{1}{1-\gamma} = \frac{\xi}{1-\xi} + \frac{\mu-\eta}{\theta[1-\eta(1-\gamma)]}\frac{\beta(\gamma-\xi)}{1-\xi+\beta(\gamma-\xi)}.$$

Meanwhile, the generating function of the waiting time under the EAS scheme is

$$W^*(z) = 1 - H_0 - H_1 + H_1\frac{\mu(1-\gamma)z}{1-[1-\mu(1-\gamma)]z}\frac{(1-\xi)[1-(1-\mu)z]}{1-[1-\mu(1-\xi)]z} + H_0\frac{[1-(1-\theta)(\eta\gamma+1-\eta)]z}{1-(1-\theta)(\eta\gamma+1-\eta)z} \times \left\{1-q+q\frac{\mu(1-\gamma)}{1-[1-\mu(1-\gamma)]z}\right\}.$$

And,

$$E(W) = H_0\left[\frac{\xi}{\mu(1-\xi)} + \frac{1}{\mu(1-\gamma)}\right] + H_1\frac{1}{\theta+\eta(1-\theta)(1-\gamma)}\left[1+\theta\frac{1-\mu(1-\gamma)}{\mu(1-\gamma)}\right],$$

where

$$H_1 = \frac{\beta(\gamma-\xi)}{1-\xi+\beta(\gamma-\xi)}, \qquad H_0 = \frac{(1-\xi)\gamma}{1-\xi+\beta(\gamma-\xi)}.$$

Certainly, when $\eta = 0$, i.e., the server can't work during the vacation period, we have $\gamma = A^*(1-\theta)$ and obtain the stationary queue length at arrival epochs and waiting time of the GI/Geo/1 queue with multiple vacations under the EAS (LAS-IA) scheme. The results are the same as those in Tian and Zhang [17]. Here, we should point out that the model considered in Tian and Zhang [17] should be a GI/Geo/1 queue with vacations under LAS-IA and not under LAS-DA.

Example 2 Classic GI/Geo/1 queue under the LAS-DA scheme.

If $\eta = \mu$ and $\theta = 1$, i.e., there is no vacation, we can obtain the classic results of the GI/Geo/1 queue under the LAS-DA scheme. Evidently, we have $c_0 = 0$ and $w_0 = 0$. Thus, the queue length distribution is

$$\pi_0 = \frac{1-\mu}{1-\mu+\mu\xi},$$
$$\pi_k = \frac{\mu\xi}{1-\mu+\mu\xi}(1-\xi)\xi^{k-1}, \quad k \ge 1.$$

And, the expected queue length at arrival epochs and the waiting time are below

$$E(L) = \frac{\xi}{1-\xi} + \frac{\mu\xi}{1-\mu+\mu\xi},$$
$$E(W) = \frac{\xi}{\mu(1-\xi)} + \frac{\xi}{1-\mu+\mu\xi} = \frac{1}{\mu}E(L).$$

Those results are the same as those in Hunter [7]. Thus, taking different values of θ or η, we can obtain the results of the special GI/Geo/1 queue with classic vacations and working vacations under EAS and LAS schemes.

Finally, we obtain the results of the GI/Geo/1 queue with working vacations under the EAS and LAS schemes. Several GI/Geo/1 models studied before are special examples of the model we consider here. Similarly important is that we also find the closed property of conditional probability for negative binomial distributions. This result makes the computation of the distribution for the waiting time easy and the expression concise and specific. In authors' opinion, such property can be used further broadly.

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