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The GI/M/1 queue with phase-type working vacations and vacation interruption

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Abstract Consider a GI/M/1 queue with phase-type working vacations and vacation interruption where the vacation time follows a phase-type distribution. The server takes the original work at the lower rate during the vacation period. And, the server can come back to the normal working level at a service completion instant if there are customers at this instant, and not accomplish a complete vacation. From the PH renewal process theory, we obtain the transition probability matrix. Using the matrix-analytic method, we obtain the steady-state distributions for the queue length at arrival epochs, and waiting time of an arbitrary customer. Meanwhile, we obtain the stochastic decomposition structures of the queue length and waiting time. Two numerical examples are presented lastly.

Keywords Working vacations · Vacation interruption · PH renewal process · Matrix-analytic approach · Stochastic decomposition

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1 Introduction

In this paper, we will consider a GI/M/1 queue with PH working vacations and vacation interruption such that the vacation time follows a Phase-type (PH) distribution. Meanwhile, the server can work at the lower rate during the vacation period, and such policy is called as working vacation. The classic M/G/1 vacation queues with various vacation policies have been well studied (see [3–5] and [6]). Two monographes of Tian and Zhang [13], Takagi [17] also collected the research results of the classic M/G/1 vacation queues. For GI/M/1 queues with server vacations, Tian et al. [14] used the matrix geometric solutions to analyze the model with multiple exponential vacations. Independently, Chatterjee and Mukherjee [2] also researched the GI/M/1 with server vacations. Subsequently, Tian and Zhang [15, 16] discussed the GI/M/1 queue with PH vacations or setup times and the discrete time GI/Geo/1 queue with server vacations, respectively. But, in these models, the server stops the original work during the vacation period, and can't come back to the normal working level until the vacation period ends.

In 2002, Servi and Finn [12] introduced the working vacation policy, in that, a customer is served at a lower rate during the vacation period. They studied an M/M/1 queue with working vacations, and analyzed a WDM optical access network using multiple wavelengths which can be reconfigured with such model. Subsequently, Wu and Takagi [18] generalized results in [12] to an M/G/1 queue with working vacations. Baba [1] considered a GI/M/1 queue with working vacations by the matrix-analytic method. Li and Tian [8, 9] analyzed two types of the discrete-time GI/Geo/1 queue with working vacations and introduced a new policy: vacation interruption in [8]. Under such a policy, the server can come back to the normal working level no matter whether the vacation ends.

We extend the results in [1] and [8] to the GI/M/1 queue with PH working vacations and vacation interruption such that the vacation time follows a Phase-type (PH) distribution. The server enters into vacation when the system becomes empty and takes service at the lower rate if there are new arrivals during the vacation period. If there are customers in the system at a service completion instant during the vacation period, the server will come back to the normal working level no matter whether the vacation ends. Otherwise, he continues the vacation. Meanwhile, to model the practical problem, the PH vacation time with multiple phases includes the exponential time with a single phase, and is applicable more broadly than an exponential time. Such a vacation queue is appropriate to model many practical problems. For examples, for optical access networks with multiple wavelengths, the data arrive randomly and are transmitted by wavelengths. In networks, wavelengths are distributed and most of them transmit the data normally, and the left wavelengths stay in Optical Network Unit (ONU) and not transmit the data. When there are more data, the left wavelengths also can be used to transmit data during a certain period, and the transmit process can be operated in multiple phases. Such policy will enable the data in networks be transmitted flexibly and decrease the costs that all wavelengths work normally. Then, the performance measures of this network operation can be presented by the results of the model we consider in this paper.

The rest of this paper is organized as follows. In Sect. 2, we firstly provide the model formulation and establish an embedded Markov chain. In Sect. 3, the rate matrix and stationary condition are given. In Sect. 4, the steady-state distribution of the embedded Markov chain at arrival epochs and the stochastic decomposition structure are presented. In Sect. 5, the waiting time distribution is obtained, and two conditional stochastic decomposition structures are given. In Sect. 6, two numerical examples are presented.

2 Model formulation and Embedded Markov chain

Consider a GI/M/1 queue such that the arrival process is a general distribution process. The server begins a vacation each time when the queue becomes empty and if there are customers arriving during the vacation period, the server continues to work at a lower rate. The working vacation period is an operation period in a lower speed. At a service completion instant during the vacation period, if there are customers in the system, the server will come back to the normal working level, i.e. vacation interruption happens. Otherwise, he continues the vacation. Meanwhile, if there is no customer when a vacation ends, the server begins another vacation, otherwise, he switches to the normal working level. Thus, the server only can go on vacations if there is no customer left in the system upon the completion of a service.

Suppose τ_n be the arrival epoch of nth customers with $\tau_0 = 0$. The inter-arrival times $\{T_n, n \geq 1\}$ are independently and identically distributed with a general distribution function, denoted by $A(t)$ with a mean $1/\lambda$ and a Laplace Stieltjes transform (LST), denoted by $a^*(s)$. Meanwhile, the vacation times follow the m-phase Phase-Type (PH) distribution with an irreducible representation $(\beta, \mathbf{S})$, where

$$\beta = (\beta_1, \beta_2, \dots, \beta_m), \quad \beta e = 1, \qquad \mathbf{Se} + \mathbf{S^0} = 0,$$

and $\mathbf{e}$ is an m-dimension row vector with all elements being 1.

The service times during the service period and the service times during the working vacation are exponentially distributed with rate μ and η, respectively.

In this model, the continuous vacation process is a PH renewal process. Denote $N(t)$ the number of renewals in $(0,t)$, and $J(t)$ the vacation phase at time t, then

$$p_{ij}(n,t) = \mathrm{P}\{N(t) = n, J(t) = j | N(0) = 0, J(0) = i\},$$

$$\mathbf{P}(n,t) = (p_{ij}(n,t))_{m\times m}, \qquad \mathbf{P}^*(z,t) = \sum_{n=0}^{+\infty} z^n \mathbf{P}(n,t).$$

From the results in [13],

$$\mathbf{P}(0,t) = \exp(\mathbf{S}t), \qquad \mathbf{P}^*(z,t) = \exp[(\mathbf{S} + z\mathbf{S}^0\beta)t].$$

Then, $p_{ij}(0,t)$ represents the probability that a vacation begins from phase i at $t = 0$, hasn't ended in $(0,t)$, and the vacation stays in phase j at t. And, note $v_i(t)$ the conditional probability that the vacation has ended at t under the condition that a vacation begins from phase i at $t = 0$. Evidently, $v_i(t) = 1 - \sum_{j=1}^m p_{ij}(0,t)$. Meanwhile, assume

$$\mathbf{q}(t) = (q_1(t), \dots, q_m(t)) = \beta \exp[(\mathbf{S} + \mathbf{S}^0\beta)t] = \beta \exp(\mathbf{S}^* t), \quad t \geq 0,$$

where $q_j(t)$ represents the probability that the vacation process stays in phase j at time t. Evidently, $\mathbf{S}^*\mathbf{e} = 1$ and $\mathbf{q}(t)\mathbf{e} = 1$.

Let $L(t)$ be the number of customers in the system at time t and $L_n = L(\tau_n - 0)$ be the number of the customers before the nth arrival. Define

$$J_n = \begin{cases} 0, & \text{the } n\text{th arrival occurs in a service period,} \\ j, & \text{the } n\text{th arrival occurs in the phase } j \text{ of working vacation period.} \end{cases}$$

Then, the process $\{(L_n, J_n), n \geq 1\}$ is an embedded Markov chain with the state space

$$\Omega = \Big\{(0, j), 1 \leq j \leq m\Big\} \cup \Big\{(k, j), k \geq 1, 0 \leq j \leq m\Big\},$$

where $(0, j)$ represents the state that there is no customer and the server stays in phase j $(1 \leq j \leq m)$ of the vacation period when a new customer arrives; $(k, 0)$ represents that a new arrival occurs in the normal service period and there are k customers in the system; $(k, j)(k \geq 1, 1 \leq j \leq m)$ represents that a new arrival occurs in phase j of the vacation period and there are k customers in the system.

In order to express the transition matrix of (L_n, J_n), let

$$\mathrm{P}_{(i,j),(k,l)} = \mathrm{P}\{L_{n+1} = k, J_{n+1} = l | L_n = i, J_n = j\}.$$

Now, we consider the transition probabilities of (L_n, J_n).

1. The transition from $(i, 0)$ to $(j, 0)$ occurs if $i+1-j$ services are completed during an inter-arrival time. Therefore, we have

$$\mathrm{P}_{(i,0),(j,0)} = \begin{cases} \int_0^\infty e^{-\mu t} \frac{(\mu t)^{i+1-j}}{(i+1-j)!} dA(t) = a_{i+1-j}, & 1 \leq j \leq i+1, \\ 0, & j \geq i+2. \end{cases} \tag{1}$$

2. The transition from $(i, 0)$ to $(0, h)$ occurs if there are $i+1$ customers are served during the normal working period, then the server enter the vacation, and the next arrival occurs in phase h of the vacation time. From the analysis of PH renewal process above, we have

$$\mathrm{P}_{(i,0),(0,h)} = \int_0^\infty \int_0^t q_h(t-u) \frac{\mu(\mu u)^i}{i!} e^{-\mu u} du dA(t), \tag{2}$$

   where $i \geq 0, 1 \leq h \leq m$.
3. The transition from (i, h) to (j, k) occurs only if $j = i+1$ arrivals occur during the same vacation time and no service is completed.

$$\mathrm{P}_{(i,h),(i+1,k)} = \int_0^\infty p_{hk}(0, t) e^{-\eta t} dA(t), \tag{3}$$

   where $i \geq 0, 1 \leq h, k \leq m$.
4. There are two possible cases to make the transition from (i, h) to $(j, 0)$. Case 1: if the residual vacation time is longer than the service time during the vacation period, during an inter-arrival time, there is one service completion by the rate η, then the server switches to the normal rate μ and $i - j$ customers are served by the

rate μ; Case 2: if the residual vacation time is not longer than the service time in the vacation period, no customer is served during the vacation time and $i+1-j$ customers are served during the service period. Then, we have

$$P_{(i,h),(j,0)} = \begin{cases} \int_0^\infty \int_0^t \eta e^{-\eta u} \sum_{k=1}^m p_{hk}(0,u) \frac{(\mu(t-u))^{i-j}}{(i-j)!} e^{-\mu(t-u)} du dA(t) \\ \quad + \int_0^\infty \int_0^t e^{-\eta u} \frac{(\mu(t-u))^{i+1-j}}{(i+1-j)!} e^{-\mu(t-u)} dv_h(u) dA(t), \\ 1 \le j \le i, \\ \int_0^\infty \int_0^t e^{-\eta u} e^{-\mu(t-u)} dv_h(u) dA(t), \quad i \ge 0, j = i+1, \end{cases} \tag{4}$$

where $1 \le h \le m$.

5. The transition from (i,h) to $(0,k)$ also may be caused by two situations: if vacation interruption happens, there is one customer served during the vacation time and i customers are served by the rate μ, then the server enters the vacation and the next arrival occurs during the vacation period; otherwise there is no customer served during the vacation time and $i+1$ customers are served during the service period. We get

$$p_{(i,h),(0,k)} = \begin{cases} \int_0^\infty \int_0^t \eta e^{-\eta u} \sum_{l=1}^m p_{hl}(0,u) \int_0^{t-u} \frac{\mu(\mu x)^{i-1}}{(i-1)!} e^{-\mu x} q_k(t-u-x) dx du dA(t) \\ \quad + \int_0^\infty \int_0^t e^{-\eta u} \int_0^{t-u} \frac{\mu(\mu x)^i}{i!} e^{-\mu x} q_k(t-u-x) dx dv_h(u) dA(t), \quad i \ge 1, \\ \int_0^\infty \int_0^t \eta e^{-\eta u} \sum_{l=1}^m p_{hl}(0,u) q_k(t) du dA(t) \\ \quad + \int_0^\infty \int_0^t e^{-\eta u} \int_0^{t-u} \mu e^{-\mu x} q_k(t-u-x) dx dv_h(u) dA(t), \quad i = 0, \end{cases} \tag{5}$$

where $1 \le h, k \le m$.

Thus, if we define

$$a_k = \int_0^\infty e^{-\mu t} \frac{(\mu t)^k}{k!} dA(t), \quad 1 \le j \le k,$$

and the m-dimensional column vectors and it follows from the relation $\mathbf{Se} + \mathbf{S^0} = 0$ that

$$\mathbf{v}_k = \mathbf{v}_k^1 + \mathbf{v}_k^0 = \int_0^\infty \int_0^t \eta \frac{(\mu(t-u))^{k-1}}{(k-1)!} e^{-\mu(t-u)} \exp((\mathbf{S} - \eta \mathbf{I})u) du dA(t) \mathbf{e}$$
$$+ \int_0^\infty \int_0^t \frac{(\mu(t-u))^k}{k!} e^{-\mu(t-u)} \exp((\mathbf{S} - \eta \mathbf{I})u) du dA(t) \mathbf{S^0}, \quad k \ge 1;$$

$$\mathbf{v}_0 = \int_0^\infty \int_0^t e^{-\mu(t-u)} \exp((\mathbf{S} - \eta \mathbf{I})u) du dA(t) \mathbf{S^0},$$

and an $m \times m$ matrix as

$$\widetilde{H}(\mathbf{S} - \eta \mathbf{I}) = \int_0^\infty \exp((\mathbf{S} - \eta \mathbf{I})t) dA(t).$$

Meanwhile, define the $m \times m$ matrix σ_k by

$$\sigma_k = \sigma_k^1 + \sigma_k^0 = \int_0^\infty \int_0^t \int_0^{t-u} \eta \frac{\mu(\mu x)^{k-1}}{(k-1)!} e^{-\mu x} \exp((\mathbf{S} - \eta \mathbf{I})u)\mathbf{e}\beta \times \exp(\mathbf{S}^*(t-u-x))dxdudA(t) + \int_0^\infty \int_0^t \int_0^{t-u} \frac{\mu(\mu x)^k}{k!} e^{-\mu x} \exp((\mathbf{S} - \eta \mathbf{I})u)\mathbf{S^0}\beta \times \exp(\mathbf{S}^*(t-u-x))dxdudA(t), \quad k \geq 1;$$

$$\sigma_0 = \sigma_0^1 + \sigma_0^0 = \int_0^\infty \int_0^t \eta \exp((\mathbf{S} - \eta \mathbf{I})u)\mathbf{e}\beta \exp(\mathbf{S}^* t)dudA(t) + \int_0^\infty \int_0^t \int_0^{t-u} \mu e^{-\mu x} \exp((\mathbf{S} - \eta \mathbf{I})u)\mathbf{S^0}\beta \times \exp(\mathbf{S}^*(t-u-x))dxdudA(t),$$

and the m-dimensional row vector $\sigma_k^{(0)}$ by

$$\sigma_k^{(0)} = \int_0^\infty \int_0^t \frac{\mu(\mu u)^k}{k!} e^{-\mu u} \beta \exp(\mathbf{S}^*(t-u))dudA(t),$$

where $\mathbf{S}^* = \mathbf{S} + \mathbf{S^0}\beta$, $\mathbf{I}$ is an identity matrix, and $\mathbf{e}$ is the row vector with all elements being 1 and proper dimension. Thus, using the lexicographical sequence for the states, the transition probability matrix of (L_n, J_n) can be written as the Block-Jacobi matrix

$$\widetilde{\mathrm{P}} = \begin{bmatrix} \mathbf{B}_{00} & \mathbf{A}_{01} & & & & \\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{A}_0 & & & \\ \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & & \\ \mathbf{B}_3 & \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{6}$$

where

$$\mathbf{B}_{00} = \sigma_0; \qquad \mathbf{A}_{01} = (\mathbf{v}_0, \widetilde{H}(\mathbf{S} - \eta \mathbf{I}));$$

$$\mathbf{A}_0 = \begin{bmatrix} a_0 & \mathbf{0} \\ \mathbf{v}_0 & \widetilde{H}(\mathbf{S} - \eta \mathbf{I}) \end{bmatrix}; \qquad \mathbf{A}_k = \begin{bmatrix} a_k & \mathbf{0} \\ \mathbf{v}_k & \mathbf{0} \end{bmatrix}, \quad k \geq 1;$$

$$\mathbf{B}_k = \begin{bmatrix} \sigma_k^{(0)} \\ \sigma_k \end{bmatrix}, \quad k \geq 1,$$

where $\mathbf{A}_{01}$ and $\mathbf{A}_k$ are $m \times (m+1)$ and $(m+1) \times (m+1)$ matrices, respectively; $\mathbf{B}_{00}$ and $\mathbf{B}_k$ are the $m \times m$ and $(m+1) \times m$ matrices, respectively. Evidently, from the expressions for $\sigma_k^{(0)}$ and σ_k, we have

$$\sigma_k^{(0)}\mathbf{e} = \int_0^\infty \int_0^t \frac{\mu(\mu u)^i}{i!} e^{-\mu u} du dA(t) = 1 - \sum_{j=0}^{k} a_j. \tag{7}$$

And,

$$\begin{aligned}
\sigma_k \mathbf{e} &= \int_0^\infty \int_0^t \int_0^{t-u} \eta \frac{\mu(\mu x)^{k-1}}{(k-1)!} e^{-\mu x} \exp((\mathbf{S} - \eta \mathbf{I})u) dx du dA(t)\mathbf{e} \\
&\quad + \int_0^\infty \int_0^t \int_0^{t-u} \frac{\mu(\mu x)^k}{k!} e^{-\mu x} \exp((\mathbf{S} - \eta \mathbf{I})u) dx du dA(t)\mathbf{S}^{\mathbf{0}} \\
&= \int_0^\infty \int_0^t \eta \left[1 - \sum_{i=0}^{k-1} \frac{(\mu(t-u))^i}{i!} e^{-\mu(t-u)}\right] \exp((\mathbf{S} - \eta \mathbf{I})u) du dA(t)\mathbf{e} \\
&\quad + \int_0^\infty \int_0^t \left[1 - \sum_{i=0}^{k} \frac{(\mu(t-u))^i}{i!} e^{-\mu(t-u)}\right] \exp((\mathbf{S} - \eta \mathbf{I})u) du dA(t)\mathbf{S}^{\mathbf{0}} \\
&= \int_0^\infty \int_0^t \exp((\mathbf{S} - \eta \mathbf{I})u) du dA(t)(\eta \mathbf{I} - \mathbf{S})\mathbf{e} - \sum_{i=0}^{k} \mathbf{v}_i \\
&= (\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I}))\mathbf{e} - \sum_{i=0}^{k} \mathbf{v}_i.
\end{aligned} \tag{8}$$

Then, we obtain

$$\mathbf{B}_k \mathbf{e} + \sum_{j=0}^{k} \mathbf{A}_j \mathbf{e} = \mathbf{e}, \quad k \geq 1; \qquad \mathbf{B}_{00}\mathbf{e} + \mathbf{A}_{01}\mathbf{e} = \mathbf{e}.$$

Thus, $\widetilde{\mathrm{P}}$ is a stochastic matrix.

3 Rate matrix and stationary condition

It is well known that, if $\rho = \lambda\mu^{-1} < 1$, the equation $z = a^*(\mu(1-z))$ has a unique solution in $(0, 1)$, denoted by ξ. To obtain the rate matrix and stationary condition, two lemmas are needed for the further discussion. Firstly, assume

$$C(\mathbf{S}) = \mathbf{S} - \eta \mathbf{I} + \mu(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I})), \qquad D(\mathbf{S}) = \widetilde{H}(\mathbf{S} - \eta \mathbf{I}) - \widetilde{H}[\mu(\widetilde{H}(\mathbf{S} - \eta \mathbf{I}) - \mathbf{I})].$$

Lemma 1 *If $\eta - \mu(1-\xi)$ is not the eigenvalue of* $\mathbf{S}$, *the matrices $\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I})$, $C(\mathbf{S})$ and $D(\mathbf{S})$ are invertible.*

Proof Note that $\mathbf{S}$ is a Metzler matrix, $\mathbf{Se} < 0$, and its eigenvalue, denoted by σ, must have negative real part. Let $\mathbf{u}$ be the corresponding eigenvector, thus $\mathbf{Su} = \sigma\mathbf{u}$, and

$$\widetilde{H}(\mathbf{S} - \eta\mathbf{I})\mathbf{u} = \int_0^\infty \exp((\mathbf{S} - \eta\mathbf{I})t)dA(t)\mathbf{u} = \int_0^\infty e^{(\sigma-\eta)t}dA(t)\mathbf{u}.$$

Thus, $\widetilde{\sigma} = \int_0^{+\infty} e^{(\sigma-\eta)t}dA(t)$ is the eigenvalue of $\widetilde{H}(\mathbf{S} - \eta\mathbf{I})$ and $\mathbf{u}$ remains to be the corresponding eigenvector. Therefore, $\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I})$ has the eigenvalue of $\xi - \widetilde{\sigma}$. Using $\sigma \neq \eta - \mu(1-\xi)$, we have $\xi - \widetilde{\sigma} \neq 0$. Thus, $\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I})$ does not have zero eigenvalue and must be invertible.

It is easy to verify that the eigenvalue of $C(\mathbf{S})$, denoted by $c(\sigma)$, is $c(\sigma) = \sigma - \eta + \mu(1-\widetilde{\sigma})$. If $\sigma - \eta \neq -\mu(1-\xi)$, $c(\sigma) \neq 0$, thus, $C(\mathbf{S})$ is invertible. Finally, for $D(\mathbf{S})$, we have

$$D(\mathbf{S})\mathbf{u} = \widetilde{H}(\mathbf{S} - \eta\mathbf{I})\mathbf{u} - \widetilde{H}[\mu(\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) - \mathbf{I})]\mathbf{u} = \left(\widetilde{\sigma} - \int_0^{+\infty} e^{-\mu(1-\widetilde{\sigma})}dA(t)\right)\mathbf{u}.$$

Thus, $d(\sigma) = \widetilde{\sigma} - \int_0^\infty e^{-\mu(1-\widetilde{\sigma})}dA(t)$ is the eigenvalue of $D(\mathbf{S})$. If $\sigma - \eta \neq -\mu(1-\xi)$, then $\xi \neq \widetilde{\sigma}$, therefore, $d(\sigma) \neq 0$ and $D(\mathbf{S})$ is invertible. □

Lemma 2 *If* $\rho = \lambda/\mu < 1$ *and* $\sigma \neq \eta - \mu(1-\xi)$, *then*

$$0 < \frac{a^*(\mu(1-\widetilde{\sigma})) - \xi}{\widetilde{\sigma} - \xi} < 1. \tag{9}$$

Proof If $\rho < 1$, $z = a^*(\mu(1-z))$ has the unique solution ξ in (0, 1). There are two cases: (i) if $0 < z < \xi$, we have $z < a^*(\mu(1-z)) < \xi$; and (ii) if $\xi < z < 1$, we have $\xi < a^*(\mu(1-z)) < z$. Now, if $\sigma < \eta - \mu(1-\xi)$, then $\widetilde{\sigma} < a^*(\mu(1-\xi)) = \xi$, and from (i), we have $\widetilde{\sigma} < a^*(\mu(1-\widetilde{\sigma})) < \xi$; If $\eta - \mu(1-\xi) < \sigma < 0$, then $1 > \widetilde{\sigma} > a^*(\mu(1-\widetilde{\sigma})) > \xi$. For both cases, we have (9). □

Theorem 1 *If* $\rho < 1$ *and* $\eta - \mu(1-\xi)$ *is not the eigenvalue of* $\mathbf{S}$, *the matrix equation* $\mathbf{R} = \sum_{k=0}^\infty \mathbf{R}^k\mathbf{A}_k$ *has the minimal nonnegative solution*

$$\mathbf{R} = \begin{bmatrix} \xi & \mathbf{0} \\ H^0 & \widetilde{H}(\mathbf{S} - \eta\mathbf{I}) \end{bmatrix},$$

where ξ *is the unique root in the range* $0 < z < 1$ *of the equation* $z = a^*(\mu(1-z))$, *and*

$$H^0 = (\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))C^{-1}(\mathbf{S})(\mathbf{S} - \eta\widetilde{H}(\mathbf{S} - \eta\mathbf{I}))\mathbf{e}.$$

Proof Because all $\mathbf{A}_k$ are lower triangular partitioned block matrices, $\mathbf{R}$ also has the same structure as

$$\mathbf{R} = \begin{bmatrix} r_{11} & \mathbf{0} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix},$$

where r_{11} is a real number, $\mathbf{R}_{21}$ is an $m \times 1$ vector, and $\mathbf{R}_{22}$ is an $m \times m$ matrix. Then, for $k \geq 1$, we have

$$\mathbf{R}^k = \begin{bmatrix} r_{11}^k & \mathbf{0} \\ \sum_{j=0}^{k-1} r_{11}^j \mathbf{R}_{22}^{k-1-j} \mathbf{R}_{21} & \mathbf{R}_{22}^k \end{bmatrix}.$$

Substituting $\mathbf{R}^k$ into the matrix equation, we obtain

$$\begin{cases} r_{11} = \sum_{k=0}^{\infty} a_k r_{11}^k = a^*(\mu(1-r_{11})), \\ \mathbf{R}_{21} = \sum_{k=0}^{+\infty} \mathbf{R}_{22}^k \mathbf{v}_k + \sum_{k=1}^{\infty} a_k \sum_{j=0}^{k-1} r_{11}^j \mathbf{R}_{22}^{k-1-j} \mathbf{R}_{21}, \\ \mathbf{R}_{22} = \widetilde{H}(\mathbf{S} - \eta \mathbf{I}). \end{cases} \tag{10}$$

As it is well known, if $\rho = \lambda/\mu < 1$, the first equation of (10) has the unique root $r_{11} = \xi$ in the range $0 < r_{11} < 1$ and $\mathbf{R}_{22} = \widetilde{H}(\mathbf{S} - \eta \mathbf{I})$. Taking $\mathbf{R}_{22}$ and r_{11} into the second equation, we have

$$\mathbf{R}_{21} = (\mathbf{I} - U(\mathbf{S}))^{-1} \sum_{k=0}^{\infty} \widetilde{H}^k (\mathbf{S} - \eta \mathbf{I}) \mathbf{v}_k, \tag{11}$$

where

$$U(\mathbf{S}) = \sum_{k=1}^{\infty} a_k \sum_{j=0}^{k-1} \xi^j \widetilde{H}^{k-1-j} (\mathbf{S} - \eta \mathbf{I}).$$

From the proof of Lemma 1, if σ^* is the eigenvalue with the maximum real part of $\mathbf{S}$, then $\sigma^* < 0$ must be real number (see [11]). Hence, the spectral radius of $\widetilde{H}(\mathbf{S} - \eta \mathbf{I})$ is $\widetilde{\sigma}^* = \int_0^{\infty} e^{(\sigma^* - \eta)t} dA(t) < 1$. Therefore, based on the results in Lemma 2,

$$SP(U(\mathbf{S})) = \sum_{k=1}^{\infty} a_k \sum_{j=0}^{k-1} \xi^{k-1-j} \widetilde{\sigma}^{*j} = \frac{a^*(\mu(1-\widetilde{\sigma}^*)) - \xi}{\widetilde{\sigma}^* - \xi} < 1.$$

Therefore, $(\mathbf{I} - U(\mathbf{S}))^{-1}$ is non-negative. And, we easily compute

$$\begin{aligned} \mathbf{I} - U(\mathbf{S}) &= \mathbf{I} - \sum_{k=0}^{\infty} a_k (\xi^k \mathbf{I} - \widetilde{H}^k (\mathbf{S} - \eta \mathbf{I}))(\xi \mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I}))^{-1} \\ &= \mathbf{I} - \int_0^{\infty} \{e^{-\mu(1-\xi)t} \mathbf{I} - \exp(-\mu(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I}))t)\} dA(t) \\ &\quad \times (\xi \mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I}))^{-1} \\ &= \{\widetilde{H}[\mu(\widetilde{H}(\mathbf{S} - \eta \mathbf{I}) - \mathbf{I})] - \widetilde{H}(\mathbf{S} - \eta \mathbf{I})\}(\xi \mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I}))^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} (\mathbf{I} - U(\mathbf{S}))^{-1} &= (\xi \mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I}))\{\widetilde{H}[\mu(\widetilde{H}(\mathbf{S} - \eta \mathbf{I}) - \mathbf{I})] - \widetilde{H}(\mathbf{S} - \eta \mathbf{I})\}^{-1} \\ &= -(\xi \mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I})) D^{-1}(\mathbf{S}). \end{aligned}$$

And, it follows from the relation $\mathbf{S}^0 = -\mathbf{S}\mathbf{e}$ that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \widetilde{H}^k(\mathbf{S}-\eta\mathbf{I})\mathbf{v}_k \\
&\quad = \int_0^\infty \int_0^t \exp\{-\mu(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))(t-u)\}\exp((\mathbf{S}-\eta\mathbf{I})u)du\,dA(t) \\
&\qquad \times (\mathbf{S^0}+\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I})\mathbf{e}) \\
&\quad = \int_0^\infty \int_0^t \exp\{-\mu(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))t\}\exp(C(\mathbf{S})u)du\,dA(t)(\mathbf{S^0}+\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I})\mathbf{e}) \\
&\quad = \int_0^\infty \{\exp((\mathbf{S}-\eta\mathbf{I})t)-\exp(-\mu(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))t)\}dA(t)C^{-1}(\mathbf{S}) \\
&\qquad \times (\mathbf{S^0}+\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I})\mathbf{e}) \\
&\quad = \{\widetilde{H}[\mu(\widetilde{H}(\mathbf{S}-\eta\mathbf{I})-\mathbf{I})]-\widetilde{H}(\mathbf{S}-\eta\mathbf{I})\}C^{-1}(\mathbf{S})(\mathbf{S}-\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\mathbf{e} \\
&\quad = -D(\mathbf{S})C^{-1}(\mathbf{S})(\mathbf{S}-\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\mathbf{e}.
\end{aligned}$$

Substituting the above equations into (11), we finally obtain the expression for $\mathbf{R}$. □

Theorem 2 *The Markov chain* $\widetilde{\mathrm{P}}$ *is positive recurrent if and only if* $\rho < 1$ *and* $\theta > 0$.

Proof Based on Theorem 1.5.1 of Neuts [10], the Markov chain $\widetilde{\mathrm{P}}$ is positive recurrent if and only if the spectral radius $SP(\mathbf{R})$ of the rate matrix $\mathbf{R}$ is less than 1, and the matrix

$$B[\mathbf{R}] = \begin{bmatrix} \mathbf{B}_{00} & \mathbf{A}_{01} \\ \sum_{k=1}^{\infty}\mathbf{R}^{k-1}\mathbf{B}_k & \sum_{k=1}^{\infty}\mathbf{R}^{k-1}\mathbf{A}_k \end{bmatrix}$$

has a positive left invariant vector, where $B[\mathbf{R}]$ is a $(2m+1)\times(2m+1)$ square matrix and

$$\mathbf{R}^{-1} = \begin{bmatrix} \xi^{-1} & \mathbf{0} \\ -\xi^{-1}\widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I}))H^0 & \widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I})) \end{bmatrix}.$$

Thus,

$$\sum_{k=1}^{\infty}\mathbf{R}^{k-1}\mathbf{A}_k = \mathbf{R}^{-1}(\mathbf{R}-\mathbf{A}_0) = \mathbf{I}-\mathbf{R}^{-1}\mathbf{A}_0 = \begin{bmatrix} 1-\frac{a_0}{\xi} & \mathbf{0} \\ \widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I}))(\frac{a_0}{\xi}H^0-\mathbf{v}_0) & \mathbf{0} \end{bmatrix}.$$

$B[\mathbf{R}]$ can be rewritten as

$$B[\mathbf{R}] = \begin{bmatrix} \sigma_0 & \mathbf{v}_0 & \widetilde{H}(\mathbf{S}-\eta\mathbf{I})) \\ \Delta_1 & 1-\frac{a_0}{\xi} & \mathbf{0} \\ \Delta_2 & \widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I}))(\frac{a_0}{\xi}H^0-\mathbf{v}_0) & \mathbf{0} \end{bmatrix}, \tag{12}$$

where Δ_1 is an m-dimensional row vector and Δ_2 is an $m \times m$ matrix, and

$$\Delta_1 = \sum_{k=1}^{\infty} \xi^{k-1} \sigma_k^{(0)};$$

$$\Delta_2 = \sum_{k=1}^{\infty} \widetilde{H}^{k-1} (\mathbf{S} - \eta \mathbf{I})) \sigma_k$$
$$+ \sum_{k=1}^{\infty} (\widetilde{H}^{k-1} (\mathbf{S} - \eta \mathbf{I})) - \xi^{k-1} \mathbf{I})(\widetilde{H}(\mathbf{S} - \eta \mathbf{I})) - \xi \mathbf{I})^{-1} H^0 \sigma_k^{(0)}.$$

Evidently, from the expression for $\mathbf{R}$, SP$(R) < 1$, thus, $B[\mathbf{R}]$ is a stochastic matrix. Δ_1 and Δ_2 don't have zero element. $B[\mathbf{R}]$ is a normal stochastic matrix, thus, there is the positive left invariant eigenvector of $B[\mathbf{R}]$ (see [7]). Then, the Markov chain $\widetilde{\mathrm{P}}$ is positive recurrent. □

4 Steady-state queue length distribution

If $\rho < 1$, let (L, J) be the stationary limit of the process (L_n, J_n). Let

$$\pi_0 = (\pi_{01}, \pi_{02}, \ldots, \pi_{0m}); \pi_k = (\pi_{k0}, \pi_{k1}, \pi_{k2}, \ldots, \pi_{km}), \quad k \geq 1$$
$$\pi_{kj} = \mathrm{P}\{L = k, J = j\} = \lim_{n \to \infty} \mathrm{P}\{L_n = k, J_n = j\}, \quad (k, j) \in \Omega.$$

To obtain the stationary probability distribution, a lemma is needed.

Lemma 3 *If $\rho < 1$, $\Delta = \sigma_0 + H^0 \Delta_1 + \widetilde{H}(\mathbf{S} - \eta \mathbf{I}))\Delta_2$ is the normal stochastic matrix.*

Proof If $\Delta \mathbf{e} = \mathbf{e}$, Δ is a normal stochastic matrix. With the relationship $\mathbf{S}^0 = -\mathbf{S}\mathbf{e}$, and (7) and (8), we have

$$\Delta_1 \mathbf{e} = \sum_{k=1}^{\infty} \xi^{k-1} \left[1 - \sum_{j=0}^{k} a_j \right] = \frac{a_0}{\xi};$$

$$\Delta_2 \mathbf{e} = \sum_{k=1}^{\infty} \widetilde{H}^{k-1} (\mathbf{S} - \eta \mathbf{I}) \left[(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta \mathbf{I})) \mathbf{e} - \sum_{i=0}^{k} \mathbf{v}_i \right]$$
$$+ \sum_{k=1}^{\infty} (\widetilde{H}^{k-1} (\mathbf{S} - \eta \mathbf{I})) - \xi^{k-1} \mathbf{I}) C^{-1} (\mathbf{S}) (\mathbf{S}^0 + \eta \widetilde{H} (\mathbf{S} - \eta \mathbf{I})) \mathbf{e}) \left[1 - \sum_{j=0}^{k} a_j \right].$$

Then, we can compute

$$\sum_{k=1}^{\infty}\widetilde{H}^{k-1}(\mathbf{S}-\eta\mathbf{I})\left[(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\mathbf{e}-\sum_{i=0}^{k}\mathbf{v}_i\right]=\mathbf{e}-\sum_{k=1}^{\infty}\widetilde{H}^{k-1}(\mathbf{S}-\eta\mathbf{I})\sum_{i=0}^{k}\mathbf{v}_i$$

$$=\mathbf{e}-\left[\sum_{k=0}^{\infty}\widetilde{H}^{k}(\mathbf{S}-\eta\mathbf{I})\sum_{i=0}^{k}\mathbf{v}_i+\sum_{k=0}^{\infty}\widetilde{H}^{k}(\mathbf{S}-\eta\mathbf{I})\mathbf{v}_{k+1}\right]$$

$$=\mathbf{e}+\widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I})\mathbf{v}_0-\widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I})(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}\sum_{k=0}^{\infty}\widetilde{H}^{k}(\mathbf{S}-\eta\mathbf{I})\mathbf{v}_k$$

$$=\mathbf{e}+\widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I})\mathbf{v}_0-[\mathbf{I}-\widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I})\widetilde{H}(\mu(\widetilde{H}(\mathbf{S}-\eta\mathbf{I})-\mathbf{I}))]$$
$$\times(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}C^{-1}(\mathbf{S})(\mathbf{S}^0+\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I})\mathbf{e}).$$

Similarly,

$$\sum_{k=1}^{\infty}(\widetilde{H}^{k-1}(\mathbf{S}-\eta\mathbf{I}))-\xi^{k-1}\mathbf{I})C^{-1}(\mathbf{S})(\mathbf{S}^0+\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I})\mathbf{e})\left[1-\sum_{j=0}^{k}a_j\right]$$

$$=-\frac{a_0}{\xi}\widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I})H^0+[\mathbf{I}-\widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I})\widetilde{H}(\mu(\widetilde{H}(\mathbf{S}-\eta\mathbf{I})-\mathbf{I}))]$$
$$\times(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}C^{-1}(\mathbf{S})(\mathbf{S}^0+\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I})\mathbf{e}).$$

Thus,

$$\Delta_2\mathbf{e}=\mathbf{e}-\widetilde{H}^{-1}(\mathbf{S}-\eta\mathbf{I})\left(\frac{a_0}{\xi}H^0-\mathbf{v}_0\right).$$

From (8), we have

$$\Delta\mathbf{e}=\sigma_0\mathbf{e}+H^0\Delta_1\mathbf{e}+\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\Delta_2\mathbf{e}=\sigma_0\mathbf{e}+\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\mathbf{e}+\mathbf{v}_0=\mathbf{e}. \qquad \square$$

Theorem 3 *If $\rho<1$, the stationary probability distribution of (L,J) is*

$$\begin{cases}\pi_0=K\widetilde{\pi},\\ \pi_k=K\widetilde{\pi}([\xi^k\mathbf{I}-\widetilde{H}^k(\mathbf{S}-\eta\mathbf{I})]C^{-1}(\mathbf{S})(\mathbf{S}-\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\mathbf{e},\widetilde{H}^k(\mathbf{S}-\eta\mathbf{I}))\quad k\geq 1,\end{cases}\tag{13}$$

where $\widetilde{\pi}$ is the invariant vector of Δ, and satisfies

$$\widetilde{\pi}\Delta=\widetilde{\pi},\qquad \widetilde{\pi}\mathbf{e}=1,$$

and,

$$K=(1-\xi)\{\widetilde{\pi}(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}((1-\xi)\mathbf{I}$$
$$+(\xi\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))C^{-1}(\mathbf{S})(\mathbf{S}-\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I})))\mathbf{e}\}^{-1}.$$

Proof Using Theorem 1.5.1 of Neuts [10], we have

$$\pi_k=\pi_1\mathbf{R}^{k-1},\quad k\geq 1.\tag{14}$$

Meanwhile, $(2m+1)$-dimensional row vector (π_0, π_1) is the positive left invariant vector of $B[\mathbf{R}]$. We can rewrite (π_0, π_1) as

$$(\pi_0, \pi_1) = (\pi_0, \pi_{10}, (\pi_{11}, \ldots, \pi_{1m})).$$

From the expression for $B[\mathbf{R}]$ in (12), $(\pi_0, \pi_1)B[\mathbf{R}] = (\pi_0, \pi_1)$ can be written as

$$\begin{cases} \pi_0\sigma_0 + \pi_{10}\Delta_1 + (\pi_{11}, \ldots, \pi_{1m})\Delta_2 = \pi_0, \\ \pi_0\mathbf{v}_0 + \pi_{10}(1 - \frac{a_0}{\xi}) + (\pi_{11}, \ldots, \pi_{1m})\widetilde{H}^{-1}(\mathbf{S} - \eta\mathbf{I})(\frac{a_0}{\xi}H^0 - \mathbf{v}_0) = \pi_{10}, \\ \pi_0\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) = (\pi_{11}, \ldots, \pi_{1m}). \end{cases}$$

Thus, we obtain

$$\begin{cases} \pi_0(\sigma_0 + H^0\Delta_1 + \widetilde{H}(\mathbf{S} - \eta\mathbf{I})\Delta_2) = \pi_0, \\ \pi_{10} = \pi_0 H^0, \\ (\pi_{11}, \ldots, \pi_{1m}) = \pi_0\widetilde{H}(\mathbf{S} - \eta\mathbf{I}). \end{cases}$$

From Lemma 3, we have $\pi_0 = K\widetilde{\pi}$, and $\widetilde{\pi}$ is the positive probability vector of the normal stochastic matrix Δ. Then,

$$\pi_0 = K\widetilde{\pi}, \quad \pi_1 = K\widetilde{\pi}(H^0, \widetilde{H}(\mathbf{S} - \eta\mathbf{I})).$$

Substituting π_1 into (14), we easily obtain the expressions for π_k. And K can be determined by the normalization condition. □

Thus, we obtain the distribution of the queue length L at the arrival epochs.

$$\mathrm{P}\{L = 0\} = \pi_0\mathbf{e} = K\widetilde{\pi}\mathbf{e},$$

$$\mathrm{P}\{L = k\} = \pi_k\mathbf{e} = K\widetilde{\pi}\left[\widetilde{H}^k(\mathbf{S} - \eta\mathbf{I})\mathbf{e} + \sum_{j=0}^{k-1}\xi^j\widetilde{H}^{k-1-j}(\mathbf{S} - \eta\mathbf{I})H^0\right], \quad K \geq 1.$$

Meanwhile, we derive the generating function of L which has the stochastic decomposition expression in the theorem below.

Theorem 4 *If $\rho < 1$ and $\mu > \eta$, the stationary queue length L can be decomposed into the sum of two independent random variables: $L = L_0 + L_d$, where L_0 is the stationary queue length of a classical* GI/M/1 *queue without vacation, and follows a geometric distribution with parameter* $1 - \xi$; *Additional queue length L_d follows a discrete* PH *distribution with probability generating function (PGF):*

$$L_d(z) = \gamma_{m+1} + z\gamma(\mathbf{I} - z\widetilde{H}(\mathbf{S} - \eta\mathbf{I}))^{-1}(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))\mathbf{e},$$

where γ is the m-dimensional vector, $\gamma_{m+1} = 1 - \gamma\mathbf{e}$ and

$$\gamma = (\mu - \eta)K^*\widetilde{\pi}(\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) - \xi\mathbf{I})C^{-1}(\mathbf{S}),$$

$$K^* = \{\widetilde{\pi}(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))^{-1}((1 - \xi)\mathbf{I} + (\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))C^{-1}(\mathbf{S})(\mathbf{S} - \eta\widetilde{H} \times (\mathbf{S} - \eta\mathbf{I})))\mathbf{e}\}^{-1}.$$

Proof From the probability expression for L above, the probability generating function of L is as follows

$$\begin{aligned}
L(z) &= \sum_{k=0}^{\infty} z^k \pi_k \mathbf{e} + \sum_{k=1}^{\infty} z^k \pi_{k0} \\
&= K\widetilde{\pi}\left[\sum_{k=0}^{\infty} z^k \widetilde{H}^k(\mathbf{S}-\eta\mathbf{I})\mathbf{e} + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \xi^j \widetilde{H}^{k-1-j}(\mathbf{S}-\eta\mathbf{I}) z^k H^0\right] \\
&= \frac{1-\xi}{1-\xi z} K^*\widetilde{\pi}[\mathbf{I} + z(\mu-\eta)(\widetilde{H}(\mathbf{S}-\eta\mathbf{I}) - \xi\mathbf{I})(\mathbf{I} - z\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1} \\
&\quad \times (\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))C^{-1}(\mathbf{S})]\mathbf{e} \\
&= L_0(z)[\gamma_{m+1} + z\gamma(\mathbf{I} - z\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}(\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\mathbf{e}] \\
&= L_0(z) L_d(z).
\end{aligned} \tag{15}$$

In the equation above, we easily verify $\gamma\mathbf{e} + \gamma_{m+1} = 1$, i.e.,

$$\begin{aligned}
&K^*\widetilde{\pi}[\mathbf{I} + (\mu-\eta)(\widetilde{H}(\mathbf{S}-\eta\mathbf{I}) - \xi\mathbf{I})C^{-1}(\mathbf{S})]\mathbf{e} - 1 \\
&\quad = K^*\widetilde{\pi}(\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}[(\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))(C(\mathbf{S}) + (\mu-\eta)(\widetilde{H}(\mathbf{S}-\eta\mathbf{I}) - \xi\mathbf{I}) \\
&\qquad - (1-\xi)C(\mathbf{S}) + (\widetilde{H}(\mathbf{S}-\eta\mathbf{I}) - \xi\mathbf{I})(\mathbf{S} - \eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))]C^{-1}(\mathbf{S})\mathbf{e} \\
&\quad = K^*\widetilde{\pi}(\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))[(\xi\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))C(\mathbf{S}) + (\widetilde{H}(\mathbf{S}-\eta\mathbf{I}) - \xi\mathbf{I})C(\mathbf{S})] \\
&\qquad \times C^{-1}(\mathbf{S})\mathbf{e} = 0.
\end{aligned}$$

Thus, the additional delay L_d follows a PH distribution with representations $(\gamma, \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))$. □

Thus, we easily obtain

$$\begin{aligned}
E(L) &= \frac{\xi}{1-\xi} + \gamma(\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}\mathbf{e} \\
&= \frac{\xi}{1-\xi} + (\mu-\eta)K^*\widetilde{\pi}(\widetilde{H}(\mathbf{S}-\eta\mathbf{I}) - \xi\mathbf{I})C^{-1}(\mathbf{S})(\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\mathbf{e}.
\end{aligned}$$

Meanwhile, the state probabilities of a server in steady-state are

$$\begin{aligned}
p_v &= \mathrm{P}\{\text{the arrival occurs during the vacation period}\} = \mathrm{P}\{J \neq 0\} \\
&= K\widetilde{\pi}(\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}\mathbf{e} = K^*\widetilde{\pi}(1-\xi)(\mathbf{I} - \widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}\mathbf{e}, \\
p_b &= \mathrm{P}\{J = 0\} = \sum_{k=1}^{\infty} \pi_{k0} = 1 - p_v.
\end{aligned} \tag{16}$$

Remark 1 If $\eta = 0$, i.e., the server doesn't take service in the vacation period, the results are the same with the classical results in the GI/M/1 queue with PH multiple vacations (see in [13]).

5 Waiting time of an arbitrary customer

In this section, we assume that the service discipline is first-come first-served. Let W and $W^*(s)$ be the steady-state waiting time and its LST, respectively.

Firstly, let H_1 be the probability that the server is in the service period when the new customer arrives, and H_0 be the probability that the server is in the vacation period and the new customer should wait when he arrives. We can easily compute

$$H_1 = K^*\widetilde{\pi}(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))^{-1}(\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))C^{-1}(\mathbf{S})(\mathbf{S} - \eta\widetilde{H}(\mathbf{S} - \eta\mathbf{I}))\mathbf{e},$$
$$H_0 = K^*\widetilde{\pi}(1 - \xi)(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))^{-1}\widetilde{H}(\mathbf{S} - \eta\mathbf{I})\mathbf{e}.$$

Theorem 5 *If $\rho < 1$, the LST of stationary waiting time W is*

$$\begin{aligned} W^*(s) = {} & 1 - H_0 - H_1 + H_1\frac{(\mu + s)(1 - \xi)}{s + \mu(1 - \xi)}\gamma^*(s\mathbf{I} - \mu(\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) - \mathbf{I}))^{-1} \\ & \times \mu(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))\mathbf{e} \\ & + H_0\{\alpha^*_{m+1} + \alpha^*(s\mathbf{I} - \mu(\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) - \mathbf{I}))^{-1}\mu(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))\mathbf{e}\} \\ & \times \beta^*(s\mathbf{I} - (\mathbf{S} - \eta\mathbf{I}))^{-1}(\eta\mathbf{I} - \mathbf{S})\mathbf{e}, \end{aligned} \tag{17}$$

where γ^*, α^* and β^* are the m-dimensional row vectors, α^*_{m+1} is the real number, and

$$\gamma^* = \frac{K^*\widetilde{\pi}(\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))C^{-1}(\mathbf{S})(\mathbf{S} - \eta\widetilde{H}(\mathbf{S} - \eta\mathbf{I}))}{K^*\widetilde{\pi}(\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))C^{-1}(\mathbf{S})(\mathbf{S} - \eta\widetilde{H}(\mathbf{S} - \eta\mathbf{I}))\mathbf{e}};$$
$$\alpha^* = \frac{K^*\widetilde{\pi}(1 - \xi)\widetilde{H}(\mathbf{S} - \eta\mathbf{I})(\eta\mathbf{I} - \mathbf{S})^{-1}(\eta\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) - \mathbf{S})}{K^*\widetilde{\pi}(1 - \xi)\widetilde{H}(\mathbf{S} - \eta\mathbf{I})\mathbf{e}};$$
$$\beta^* = \frac{K^*\widetilde{\pi}(1 - \xi)\widetilde{H}(\mathbf{S} - \eta\mathbf{I})}{K^*\widetilde{\pi}(1 - \xi)\widetilde{H}(\mathbf{S} - \eta\mathbf{I})\mathbf{e}}$$

and $\alpha^*_{m+1} = 1 - \alpha^*\mathbf{e}$.

Proof Firstly, we obtain the probability that a new customer shouldn't wait

$$\mathrm{P}\{W = 0\} = \pi_0\mathbf{e} = K^*\widetilde{\pi}(1 - \xi)(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))\mathbf{e} = 1 - H_0 - H_1.$$

When a new customer arrives, if there are k customers and the server is in the normal working period, the waiting time equals k service times by the rate μ. Then, we easily have

$$\begin{aligned} \sum_{k=1}^{\infty}\pi_{k0}W^*_{k0}(s) &= \sum_{k=1}^{\infty}\pi_{k0}\left(\frac{\mu}{\mu + s}\right)^k \\ &= \frac{(\mu + s)(1 - \xi)}{s + \mu(1 - \xi)}K^*\widetilde{\pi}(\xi\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I}))C^{-1}(\mathbf{S})(\mathbf{S} - \eta\widetilde{H}(\mathbf{S} - \eta\mathbf{I})) \end{aligned}$$

$$\times (s\mathbf{I} - \mu(\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) - \mathbf{I}))^{-1}\mu(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I})\mathbf{e}$$
$$= H_1 \frac{(\mu + s)(1 - \xi)}{s + \mu(1 - \xi)} \gamma^*(s\mathbf{I} - \mu(\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) - \mathbf{I}))^{-1}$$
$$\times \mu(\mathbf{I} - \widetilde{H}(\mathbf{S} - \eta\mathbf{I})\mathbf{e}. \tag{18}$$

And, it is obvious that the latter part of equation above is the LST of a PH distribution with an irreducible representation $(\gamma^*, \mu(\widetilde{H}(\mathbf{S} - \eta\mathbf{I}) - \mathbf{I}))$.

Then, we consider the situation that the system state is (k, h) when the new customer arrives. Denote V and B_2 be the residual vacation time and the vacation service time, respectively. Then,

$$\mathrm{P}\{B_2 < t, V > B_2|h\} = \int_0^t \eta e^{-\eta x} \sum_{j=1}^m p_{hj}(0, x)dx.$$

We obtain the LST of joint probability of the service time during the vacation period and the condition $\{V > B_2\}$.

$$B_2{}^{(v)}(s) = \int_0^\infty e^{-st} d\mathrm{P}\{B_2 < t, V > B_2\}$$
$$= \int_0^\infty \eta \exp\{-(s\mathbf{I} - (\mathbf{S} - \eta\mathbf{I}))t\}dt\mathbf{e} = \eta(s\mathbf{I} - (\mathbf{S} - \eta\mathbf{I}))^{-1}\mathbf{e}.$$

Similarly, we have

$$\mathrm{P}\{V < t, V < B_2|h\} = \int_0^t e^{-\eta x} dv_h(x).$$

And, the LST of vacation time under the condition $\{V < B_2\}$ is

$$V^{(B)}(s) = \int_0^\infty \exp\{-(s\mathbf{I} - (\mathbf{S} - \eta\mathbf{I}))t\}dt\mathbf{S}^0 = -(s\mathbf{I} - (\mathbf{S} - \eta\mathbf{I}))^{-1}\mathbf{S}\mathbf{e}.$$

Evidently, $B_2^{(v)}(s)$ and $V^{(B)}(s)$ are m-dimensional column vectors.

There are two cases for the waiting time of the new customer if the state is (k, h) when he arrives. Case 1: If the residual vacation time is longer than one service time by the rate η, i.e., $\{V > B_2|h\}$, after a service completion in the vacation period, vacation interruption happens and the server comes back to the normal working level rather than keeping on the vacation. Thus, the waiting time is the sum of one service time by the rate η under the condition $\{V > B_2|h\}$ and $k - 1$ service times by the rate μ. Case 2: If the residual vacation time is not longer than the service time in the vacation period, i.e., $\{V < B_2|h\}$, when a vacation ends, the server has not completed a service during the vacation and comes back to the normal level. Therefore, the waiting time equals the sum of the residual vacation time under the condition $\{V < B_2|h\}$ and k service times by the rate μ. Thus, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\pi_k\widetilde{\mathbf{W}}_k^*(s)\\
&\quad=\sum_{k=1}^{\infty}\pi_k\left[B_2^{(v)}(s)\left(\frac{\mu}{\mu+s}\right)^{k-1}+V^{(B)}(s)\left(\frac{\mu}{\mu+s}\right)^{k}\right]\\
&\quad=K\widetilde{\pi}\sum_{k=1}^{\infty}\widetilde{H}^k(\mathbf{S}-\eta\mathbf{I})\left[\eta\mathbf{I}\left(\frac{\mu}{\mu+s}\right)^{k-1}-\mathbf{S}\left(\frac{\mu}{\mu+s}\right)^{k}\right](s\mathbf{I}-(\mathbf{S}-\eta\mathbf{I}))^{-1}\mathbf{e}\\
&\quad=K\widetilde{\pi}\widetilde{H}^k(\mathbf{S}-\eta\mathbf{I})[\eta\mathbf{I}+\mu(\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I})-\mathbf{S})(s\mathbf{I}-\mu(\widetilde{H}(\mathbf{S}-\eta\mathbf{I})-\mathbf{I}))^{-1}]\\
&\qquad\times(s\mathbf{I}-(\mathbf{S}-\eta\mathbf{I}))^{-1}\mathbf{e}\\
&\quad=H_1\{\alpha_{m+1}^*+\alpha^*(s\mathbf{I}-\mu(\widetilde{H}(\mathbf{S}-\eta\mathbf{I})-\mathbf{I}))^{-1}\mu(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\mathbf{e}\}\\
&\qquad\times\beta^*(s\mathbf{I}-(\mathbf{S}-\eta\mathbf{I}))^{-1}(\eta\mathbf{I}-\mathbf{S})\mathbf{e},
\end{aligned}\tag{19}
$$

where α^*, α_{m+1}^* and β^* are defined as above and $\widetilde{\mathbf{W}}_k^*(s)$ is an m-dimensional column vector. And, we easily have

$$
W^*(s)=\pi_0\mathbf{e}+\sum_{k=1}^{\infty}\pi_{k0}W_{k0}^*(s)+\sum_{k=1}^{+\infty}\pi_k\widetilde{\mathbf{W}}_k^*(s).
$$

From (18) and (19), we have the result in Theorem 5. □

With the structure in Theorem 5, we can easily get the expected waiting time

$$
\begin{aligned}
E(W)=K^*\widetilde{\pi}(\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))^{-1}\frac{1}{\mu}\Bigg[&(\xi\mathbf{I}-\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))C^{-1}(\mathbf{S})\\
&\times(\mathbf{S}-\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))\frac{\xi}{1-\xi}\\
&-\mu(1-\xi)\widetilde{H}(\mathbf{S}-\eta\mathbf{I})(\mathbf{S}-\eta\mathbf{I})^{-1}+[\mathbf{I}+\mu(1-\xi)(\mathbf{S}-\eta\mathbf{I})^{-1}]\\
&\times(\mathbf{S}-\eta\widetilde{H}(\mathbf{S}-\eta\mathbf{I}))C^{-1}(\mathbf{S})\Bigg]\mathbf{e}.
\end{aligned}
$$

For the waiting time, we have two theorems below to present the conditional stochastic decomposition structures. First, denoting by W_1 the conditional waiting time when the server is in the busy period, i.e., $J=0$, we obtain

Theorem 6 *W_1 can be decomposed into the sum of two independent random variables*: $W_1=W_0+W_d$, *where W_0 is the waiting time of a classical* GI/M/1 *queue without vacations, and follows a modified exponential distribution with parameter ξ; Additional queue length L_d follows a PH distribution with an irreducible representation* $(\gamma^*,\mu(\widetilde{H}(\mathbf{S}-\eta\mathbf{I})-\mathbf{I}))$.

Proof From the definition of W_1, its LST is as follows

$$W_1^*(s) = \frac{1}{\mathrm{P}\{J=0\}} \sum_{k=1}^{\infty} \pi_{k0} \left(\frac{\mu}{\mu + s} \right)^k.$$

From (18), we easily obtain the results and don't explain in detail. □

Similarly, for W_2 the conditional waiting time when the server is in the vacation period and the new customer should wait, i.e., $\{J \neq 0, L > 0\}$, we also have

Theorem 7 *W_2 can be decomposed into the sum of two independent PH random variables with representations $(\alpha^*, \mu(\widetilde{H}(\mathbf{S} - \eta \mathbf{I}) - \mathbf{I}))$ and $(\beta^*, \mathbf{S} - \eta \mathbf{I})$, respectively.*

Remark 2 For (17), the steady-state waiting time of an arbitrary customer has the special probability explanation. The waiting time equals zero with the probability $1 - H_0 - H_1$; with the probability H_0, it equals the sum of one exponential random variable with the rate $\mu(1 - \xi)$ and one PH random variable with the irreducible representations $(\gamma^*, \mu(\widetilde{H}(\mathbf{S} - \eta \mathbf{I}) - \mathbf{I}))$; with the probability H_1, it equals the sum of two random variables with PH distributions with irreducible representations $(\alpha^*, \mu(\widetilde{H}(\mathbf{S} - \eta \mathbf{I}) - \mathbf{I}))$ and $(\beta^*, \mathbf{S} - \eta \mathbf{I})$, respectively.

6 Numerical examples

To demonstrate the applicability of the results we obtain above, two special examples are presented below to analyze the operating performance measures of the system.

Example 1 In Table 1, we present the stationary distributions of a special model with a two-phase PH vacation time, where some numerical parameters are given below:

$$\beta = [0.77,\ 0.230], \quad S = \begin{bmatrix} -4 & 2 \\ 0 & -3 \end{bmatrix}, \quad S^0 = [2,\ 3]^T,$$

where T is the "matrix transpose operation". Meanwhile, we assume the Poisson arrival process with rate $\lambda = 0.6$, and $\mu = 1.0$ and $\eta = 0.5$. Then a special model with the PH vacation time is established. It follows from the results in sections above that the queue length distributions are presented in Table 1. And, we can further obtain the other systems indices, including the expected queue length $E(L) = 1.6033$ and the expected waiting time $E(W) = 1.9875$.

Example 2 We demonstrate the change of the expected queue length with the vacation service rate η in the model with exponential vacation times with the rate θ in Fig. 1.

Two types of inter-arrival time distributions: (1) exponential, and (2) E_2, are considered. They all have equal mean $(= 1/\lambda)$ where $\lambda = 1.25$. Other parameters are

Table 1 The queue length distributions for Example 1

k	π_{k0}	π_{k1}	π_{k2}	$P\{L=k\}$
0	0.1876	0.1775	–	0.3651
1	0.1897	0.0221	0.0367	0.2485
2	0.1445	0.0026	0.0066	0.1537
3	0.0915	0.0003	0.0011	0.0929
4	0.0557	0.0359×10^{-3}	0.1816×10^{-3}	0.0559
5	0.0335	0.0423×10^{-4}	0.2864×10^{-4}	0.0336
6	0.0201	0.0497×10^{-5}	0.4434×10^{-5}	0.0202
7	0.0121	0.0585×10^{-6}	0.6774×10^{-6}	0.0121
8	0.0072	0.0069×10^{-6}	0.1025×10^{-6}	0.0072
9	0.0043	0.0081×10^{-7}	0.1539×10^{-7}	0.0043
Sum	0.7462	0.2025	0.0448	0.9935

taken as $\mu = 2.0$ and $\eta = 0.6$. As illustrated in Fig. 1, with the increase of the vacation service rate, $E(L)$ decreases evidently and the utilization level of the system idle time becomes larger. Meanwhile, when the service rate η approaches to μ, the vacation rate doesn't have the effect and the models become the corresponding queues without vacations. And the effects of the vacation rate θ and traffic intensity ρ also are agreement with the practical cases so that our model is reasonable with the practice usage.

7 Conclusions

In this paper, we consider a GI/M/1 queue with PH working vacations and vacation interruption. Such two policies are introduced recently and the server in the system can work at the lower speed during the vacation period and also can come back to the normal working level no matter whether the vacation ends. Meanwhile, we introduce the PH vacation time and extend the generality of queues with working vacation. Many models of the GI/M/1 vacation queues, such as the classic GI/M/1 queue with PH and exponential vacations, are the special examples of the model we consider. The steady-state results of the queue length and waiting time are obtained and the stochastic decomposition results are established. Thus, with these results, we not only establish the theoretical framework for the GI/M/1 queue with PH working vacations and vacation interruption, but also can model some practical problems in management and communications directly.

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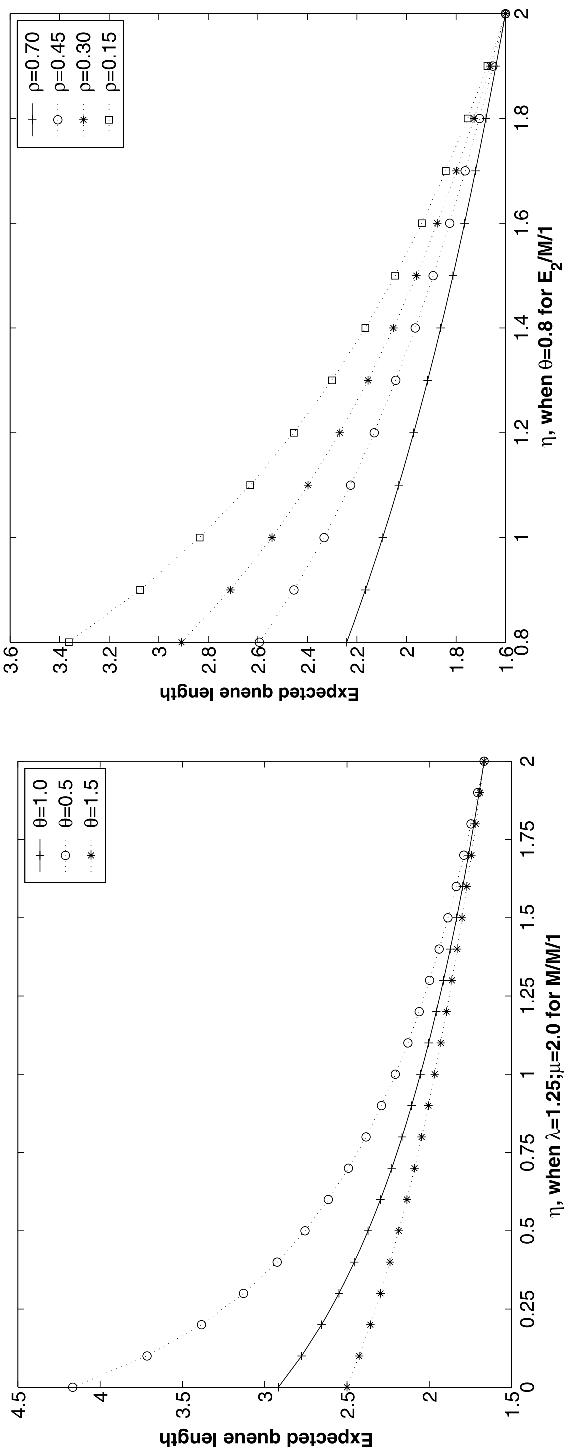

Fig. 1 Expected queue length $E(L)$ vs. η

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