# Can unpredictable risk exposure be priced?

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# Abstract

The link between beta predictability and the price of risk.

An investor who desires exposure to a certain risk factor needs to predict what next period's beta will be.

use a model to show that an ambiguity averse agent's demand is lower when betas are hard to predict, leading to a reduction in risk premiums.

test the implications for downside betas and VIX betas.

find that they have economically and statistically small prices of risk

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# 1.Introduction

- In Section 2 we develop our theoretical model to illustrate how parameter uncertainty around asset correlations affects asset prices.
- In Section 3 we find the best possible prediction between beta prediction models, and in Section 4 we compare risk premium estimates using expost realized beta specifications vis–vis premium estimates using predicted or lagged betas.
- In Section 5 we explain that beta uncertainty issues are not confined to the realm of relative downside and VIX betas, and in Section 6 we explain why risk premium estimates using ex-post realized betas may contain an upward bias.
- In Section 7 we optimize the portfolio of an investor with Constant Relative Risk Aversion (CRRA) preferences, taking into account beta uncertainty, and measure the decrease in hedge portfolio demand induced by lack of predictability, and Section 8 concludes.

## 2. Ambiguity aversion and correlation uncertainty

- This section discusses a simple model relating beta predictability to hedging demands and prices of risk. Our model consists of two ambiguity-averse agents optimizing their utility.

- We create hedging demand by exposing one agent to an exogenous shock, while the other agent is not affected. Both agents can invest in the market and a zero net supply risky asset with payoffs that are possibly correlated to the exogenous shock.

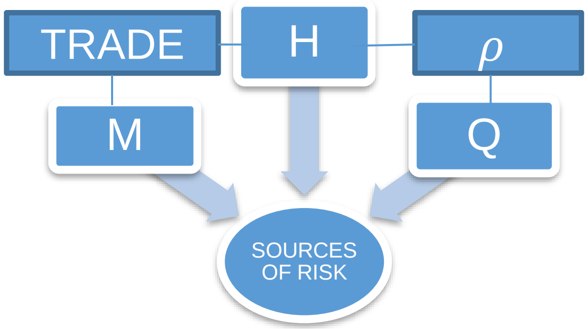
- The degree to which the risky asset hedges the exogenous shock remains uncertain.

- An agent’s ability to forecast risk exposures affects their hedging demands and therefore the price of the risky asset.

## 2.1. Market model with a zero net supply portfolio

### 2.1.1 Assets

- Three sources of risk: (the market portfolio M, hedging asset H, and exogenous risk, Q), M and H are traded. The market asset must be in positive net supply, the hedging portfolio is in zero net supply.




- Excess returns of the three risky assets

$$r_{t+1} = \begin{bmatrix} r_{M,t+1} \\ r_{H,t+1} \\ r_{Q,t+1} \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_M \\ \mu_H \\ \mu_Q \end{bmatrix}, \begin{bmatrix} \sigma_M^2 & 0 & 0 \\ 0 & \sigma_H^2 & \sigma_{QH} \\ 0 & \sigma_{QH} & \sigma_Q^2 \end{bmatrix} \right),$$

$\sigma_{QH} = \sigma_Q \sigma_H \rho$, where $\rho$ is the correlation coefficient between Q and H.

- The total market capitalization must be equal to the total traded wealth in the economy, $W_T$ , which we normalize to one.
- The expected returns of M and H will be endogenous to the model, while the expected return of Q is given.

### 2.1.2 Agents

In this model, we have two types of agents, j ∈ { A, B }, with risk aversion $\gamma_j$ where agent A is not exposed to exogenous risks, while agent B has a positive exposure to the exogenous risk.

Agents of type A and B detain all of the wealth in the economy and each own a relative share of wealth $W_j$. They must invest a portion of their wealth in each of the two traded assets: the market asset M and the hedge asset H.

Excess portfolio returns $$r_{p,t+1}^{j} = w_M^j r_{M,t+1} + w_H^j r_{H,t+1} + w_Q^j r_{Q,t+1} \quad (2)$$

Expected return $$E\left(r_{p,t+1}^{j}\right) = w_M^j \mu_M + w_H^j \mu_H + w_Q^j \mu_Q \quad (3)$$

Variance $$Var\left(r_{p,t+1}^{j}\right) = \left(w_M^j\right)^2 \sigma_M^2 + \left(w_H^j\right)^2 \sigma_H^2 + \left(w_Q^j\right)^2 \sigma_Q^2 + 2 w_H^j w_Q^j \sigma_H \sigma_Q \rho \quad (4)$$

Confidence interval $$\rho \in \left[\rho - \eta; \rho + \eta\right]$$

$w_M^j$和$w_H^j$ : 投资于每种资产的财富部分

$\mu_M$和$\mu_H$： 均衡状态下以什么价格交易

In our framework, a higher η can either represent more ambiguity aversion or a higher standard error of the estimate. Keeping ambiguity aversion constant, higher η also means higher correlation uncertainty.

In our model, each agent wants to maximize next period's expected utility, which we model using a mean-variance utility function.

Objective function for agent A:

$$\max_{\{w_M^A, w_H^A\}} U^A = w_M^A \mu_M + w_M^A \mu_H - \frac{\gamma^A}{2}\left(\left(w_M^A\right)^2 \sigma_M^2 + \left(w_H^A\right)^2 \sigma_H^2\right) \qquad (5)$$

For agent B:

$$\begin{aligned} \max_{\{w_M^B, w_H^B\}} \min_{\{\rho\}} U^B &= w_M^B \mu_M + w_H^B \mu_H + w_Q \mu_Q \\ &- \frac{\gamma^B}{2}((w_M^B)^2 \sigma_M^2 + (w_H^B)^2 \sigma_H^2 + (w_Q)^2 \sigma_Q^2 \\ &+ 2 w_H^B w_Q \sigma_H \sigma_Q \rho). \\ &s.t. \quad \rho \le \hat{\rho} + \eta \\ &\quad\quad \rho \ge \hat{\rho} - \eta. \end{aligned} \qquad (6)$$

For simplicity, since $w_Q^A = 0$, we redefine $w_Q \equiv w_Q^B$.

### 2.1.3 **Equilibrium conditions**

- Market equilibrium is achieved once both agents optimally allocate to the available assets given their current prices, $\mu_M$ and $\mu_H$, which are endogenously determined, and the remaining exogenous parameters.
- Proposition 1 . If there is no correlation uncertainty, $\hat{\rho} = \rho$ and $\eta = 0$ , then the optimal partial equilibrium demands for agents A and B are given by

$$\begin{cases} w_M^{A*} = \dfrac{\mu_M}{\gamma^A \sigma_M^2} \\ w_H^{A*} = \dfrac{\mu_H}{\gamma^A \sigma_H^2} \end{cases} \qquad (7)$$

*and for B,*

$$\begin{cases} w_M^{B*} = \dfrac{\mu_M}{\gamma^B \sigma_M^2} \\ w_H^{B*} = \dfrac{\mu_H}{\gamma^B \sigma_H^2} - \dfrac{w_Q \sigma_H \sigma_Q \rho}{\sigma_H^2}. \end{cases} \qquad (8)$$

In equilibrium, markets for both tradable assets have to clear. This means that total wealth allocated to the market portfolio has to add up to one and total wealth invested in the hedge asset should sum to zero.

$$W_A w_M^A + W_B w_M^B = W_T = 1, \tag{9}$$

and

$$W_A w_H^A + W_B w_H^B = 0. \tag{10}$$

- Proposition 2 . If there is no correlation uncertainty, $\hat{\rho} = \rho$, and $\eta = 0$ , then the market premium and hedge asset premiums are given by

$$\begin{aligned} \mu_M &= \bar{\gamma}\sigma_M^2 \\ \mu_H &= \bar{\gamma} W_B w_Q \sigma_H \sigma_Q \rho, \end{aligned} \tag{11}$$

$$\bar{\gamma} = \sum_{j \in A,B} \left( \frac{W_j}{\gamma^j} \right)^{-1}.$$

$\bar{\gamma}$ : wealth-weighted harmonic average of both agents' risk aversion coefficients.

The key component of the risk premium is the correlation between the hedge asset and the exogenous risk. In absolute terms, higher correlations result in higher premiums, while the sign of the premium depends on whether the correlation is negative or positive.

- Proposition 3 . Under ambiguity aversion, agent B's partial equilibrium demand for the hedge asset and the optimal correlation ( $\rho^*$ ) is given by

$$\rho^* = \begin{cases} \hat{\rho} + \eta, & \text{if } \hat{\rho} < -\eta \\ 0, & \text{if } \hat{\rho} - \eta \le 0 \le \hat{\rho} + \eta \\ \hat{\rho} - \eta, & \text{if } \hat{\rho} > \eta \end{cases} \tag{12}$$

$$w_H^{B*} = \begin{cases} \frac{\mu_H}{\gamma^B \sigma_H^2} - \frac{w_Q \sigma_H \sigma_Q (\hat{\rho}+\eta)}{\sigma_H^2}, & \text{if } \hat{\rho} < -\eta \\ 0, & \text{if } \hat{\rho} - \eta \le 0 \le \hat{\rho} + \eta \\ \frac{\mu_H}{\gamma^B \sigma_H^2} - \frac{w_Q \sigma_H \sigma_Q (\hat{\rho}-\eta)}{\sigma_H^2}, & \text{if } \hat{\rho} > \eta. \end{cases} \tag{13}$$

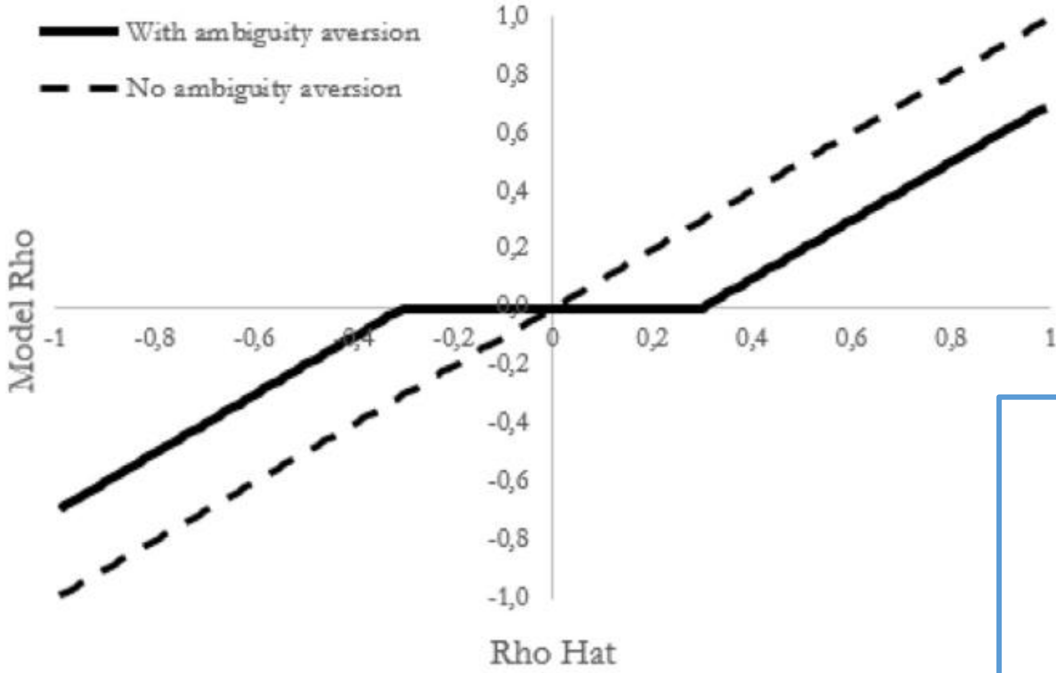
Agent A's partial equilibriums demands and agent B's demand for the market remain unchanged. Agent B's hedging demand shrinks toward zero if correlation uncertainty, η , is high.

- Proposition 4 . We insert optimal demands for both agents into the market-clearing conditions in Eq. (10) , and obtain the following equilibrium returns for the hedge asset.

$$\mu_H = \begin{cases} \bar{\gamma} W_B w_Q \sigma_H \sigma_Q (\hat{\rho} + \eta), & \text{if } \hat{\rho} < -\eta \\ 0, & \text{if } \hat{\rho} - \eta \le 0 \le \hat{\rho} + \eta \\ \bar{\gamma} W_B w_Q \sigma_H \sigma_Q (\hat{\rho} - \eta), & \text{if } \hat{\rho} > \eta. \end{cases} \tag{14}$$

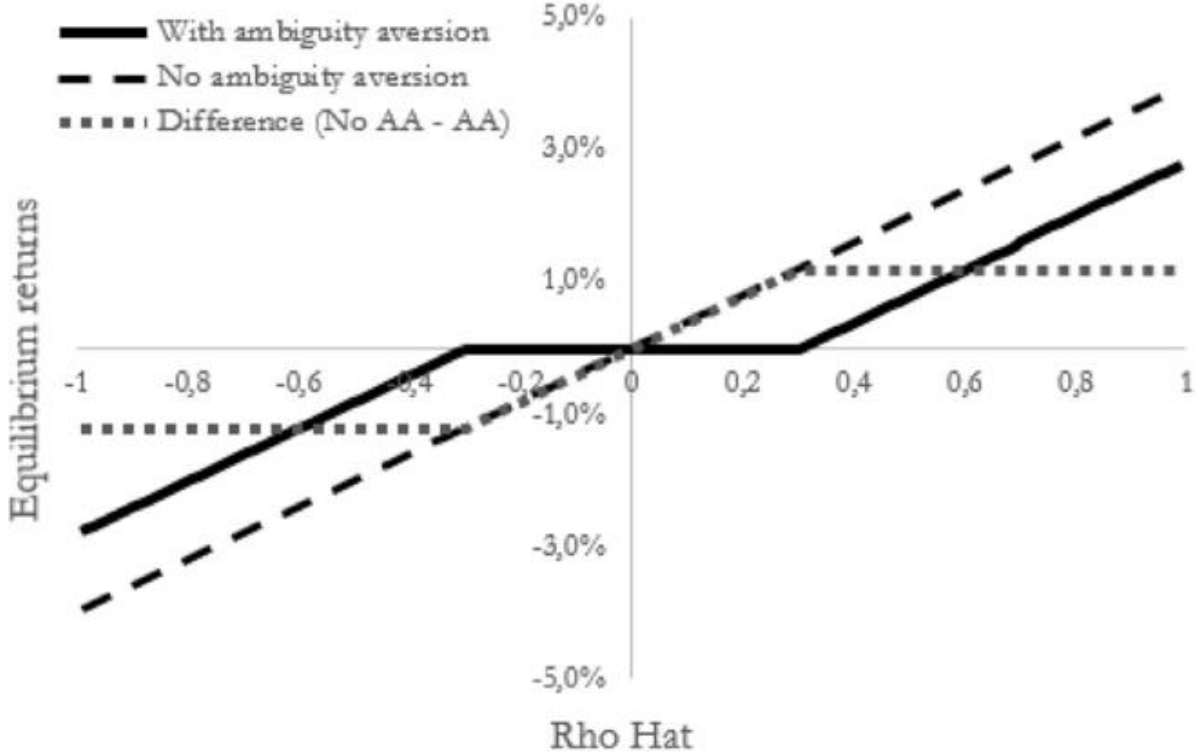
**Fig. 1.** Comparing models with and without ambiguity aversion

Panel A: $\rho$ used by agent $B$ to determine optimal demand under ambiguity aversion and no ambiguity aversion.

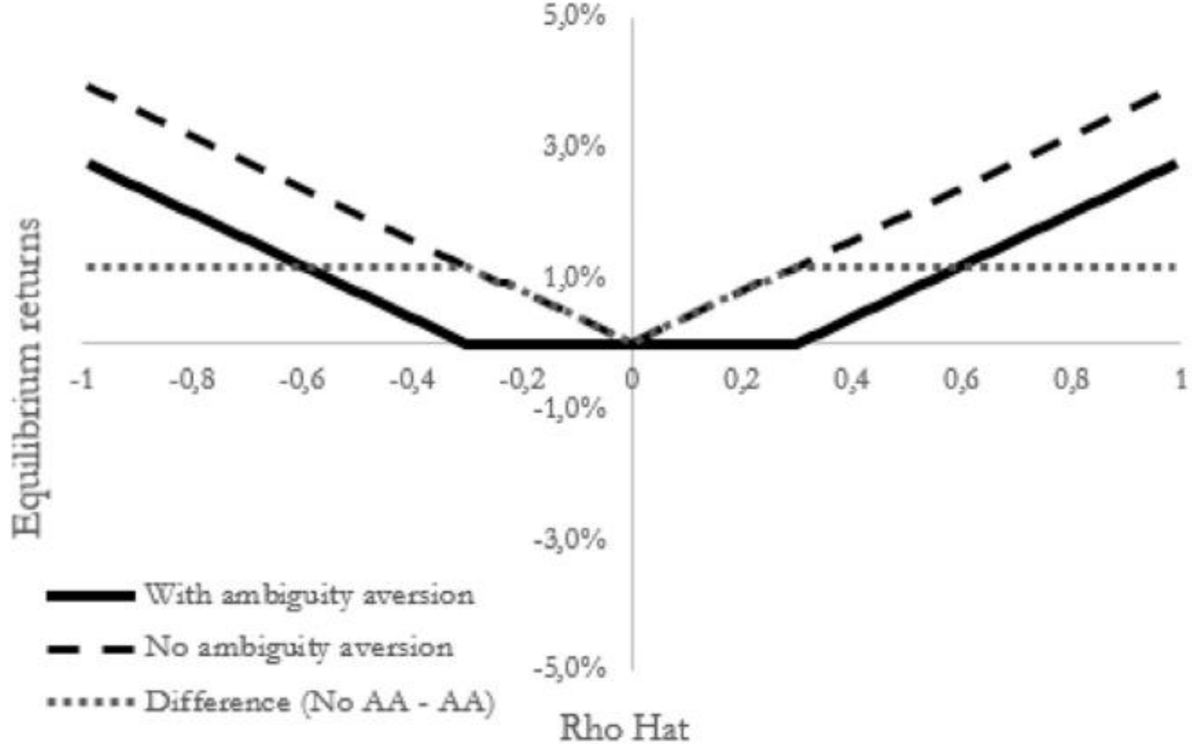



$\eta = 0.3$
$\gamma_A=4,\ \gamma_B=4$
$\sigma_H = 0.2$
$\sigma_Q = 0.2$
$w_Q=0.5$
$w_A=0.5,\ w_B=0.5$

Panel B: Equilibrium $\mu_H$ under ambiguity aversion and no ambiguity aversion



Panel C: Comparing equilibrium $\mu_H$ in absolute terms under ambiguity aversion and no ambiguity aversion.



### 3.Predicting betas

- In this section we focus on relative downside betas and VIX betas. More specifically, we forecast ex-post realized betas — which are often used in asset pricing tests but unobservable for investors when forming hedging demands.

- Use daily data over the course of a year to estimate a stock’s yearly beta at each point in time, we use daily return data because a higher sampling frequency should improve the beta estimates.

- When estimating betas, stocks with more than five missing daily returns observations over the course of the estimation window are removed from our sample.

## 3.1 Relative downside and VIX betas

- Using one lag for CAPM betas to mitigate non synchronous trading concerns $$r_{i,t} = \alpha_i + \beta_{i,1} r_{m,t} + \beta_{i,2} r_{m,t-1} + e_{i,t}$$
- Standard downside betas are calculated as the market beta .

$$\beta_i^{Downside} = \frac{Cov\left(r_{i,t}, r_{m,t} \mid r_{m,t} < \mu_m\right)}{Var\left(r_{m,t} \mid r_{m,t} < \mu_m\right)} \longrightarrow$$

estimate a stock's market beta using only days for which the market had a return below its average over the past year

$\beta^{DR}$(relative downside beta)=standard downside beta—market beta

- VIX betas ($\beta^{VIX}$ ) : regressing stock returns on daily first differences in the VIX, a volatility index based on the implied volatility on S&P 500 options

$$r_{i,t} = \alpha_i + \beta_{i,1} r_{m,t} + \beta_{i,2} r_{m,t-1} + \beta_{i,3} r_{k,t} + \beta_{i,4} r_{k,t-1} + e_{i,t}$$

$r_m$: the market return； $r_k$ : a daily change in the volatility index.

### 3.2 Predictive regressions

- We predict future betas using realized lagged betas and then use firm characteristics to improve beta forecasts.
- We run cross-sectional predictive regressions every month and report average coefficients with corresponding t-statistics in Table 1

$$\beta_{i,t+12} = \delta_{0t} + \delta_{1t}\beta_{i,t} + \delta_{2t}X_{i,t} + e_{i,t+12} \tag{18}$$

$\beta_{i,t}$: the most recently estimated beta (the lagged beta), which uses the past 12 months of data, and $X_{i,t}$ is a vector of predictive variables.

- Co-skewness (CSK):

$$CSK_i = \frac{E[(r_i - \mu_i).(r_m - \mu_m)^2]}{\sqrt{Var(r_i).Var(r_m)}},$$

$\mu_i$ and $\mu_m$ : the stock's and the market mean returns,

$Var(r_i)$and $Var\ (r_m)$: the stock and market variances over the past year using daily data.

- Idiosyncratic volatility(IV) : the volatility of the CAPM error terms:

$$\epsilon_{i,t} = r_{i,t} - \beta_i^M r_{m,t},$$

$\beta_i^M$ : the market beta of stock $i$ estimated over the past year using daily data

# Table 1 Predictive beta regressions.

Panel A: Relative downside beta

| $\beta^{DR}_{t-1}$ | IV | CSK | log(Size) | BTM | ROE | Lev. | $\beta^{DR,HF}_{t-1}$ | OI skew | Constant | Ind dummies | Adj. $R^2$ | Sample |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| −0.02 | | | | | | | | | Yes | No | 0.027 | All |
| −2.94 | | | | | | | | | | | | |
| −0.03 | 0.05 | | | | | | | | Yes | No | 0.055 | All |
| −6.98 | 4.03 | | | | | | | | | | | |
| −0.03 | 0.05 | −0.02 | | | | | | | Yes | No | 0.061 | All |
| −7.20 | 4.17 | −5.41 | | | | | | | | | | |
| −0.03 | 0.04 | −0.02 | −0.02 | −0.02 | −0.14 | 0.02 | | | Yes | No | 0.082 | All |
| −6.79 | 5.73 | −5.50 | −1.69 | − 3.15 | −2.48 | 4.02 | | | | | | |
| −0.02 | 0.04 | −0.01 | −0.02 | −0.01 | −0.08 | 0.01 | | | Yes | Yes | 0.113 | All |
| −6.74 | 5.71 | −5.82 | −1.82 | − 2.39 | −1.78 | 4.69 | | | | | | |
| 0.04 | 0.03 | 0.01 | −0.01 | −0.06 | 0.06 | 0.05 | | | Yes | Yes | 0.173 | HF |
| 3.46 | 3.18 | 1.10 | −0.62 | −3.85 | 0.29 | 2.58 | | | | | | |
| 0.03 | 0.04 | 0.01 | 0.00 | −0.06 | 0.04 | 0.05 | −0.02 | | Yes | Yes | 0.183 | HF |
| 3.43 | 4.81 | 1.07 | −0.36 | −3.96 | 0.19 | 2.76 | −2.62 | | | | | |
| 0.03 | 0.02 | 0.00 | 0.02 | −0.01 | −0.02 | 0.02 | | | Yes | Yes | 0.227 | OI |
| 4.72 | 1.20 | 0.06 | 0.93 | −1.41 | −0.50 | 1.97 | | | | | | |
| 0.02 | 0.01 | −0.01 | −0.00 | −0.00 | 0.01 | 0.02 | | −0.01 | Yes | Yes | 0.235 | OI |
| 3.49 | 0.31 | −0.96 | −0.29 | −0.13 | 0.11 | 2.46 | | −1.83 | | | | |

Panel B: VIXbeta

| $\beta^{VIX}_{t-1}$ | IV | CSK | Size | BTM | ROE | Lev. | $\beta^{VIX,HF}_{t-1}$ | OI skew | Constant | Ind dummies | Adj. $R^2$ | Sample |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.02 | | | | | | | | | Yes | No | 0.036 | All |
| 3.35 | | | | | | | | | | | | |
| 0.02 | 0.01 | | | | | | | | Yes | No | 0.053 | All |
| 3.51 | 1.18 | | | | | | | | | | | |
| 0.02 | 0.01 | 0.01 | | | | | | | Yes | No | 0.055 | All |
| *3.55* | *1.13* | *2.50* | | | | | | | | | | |
| 0.02 | 0.00 | 0.01 | −0.02 | 0.01 | −0.05 | −0.01 | | | Yes | No | 0.072 | All |
| *4.00* | *−0.78* | *2.71* | *−3.14* | *1.41* | *−0.94* | *−2.21* | | | | | | |
| 0.02 | −0.01 | 0.00 | −0.02 | 0.00 | −0.08 | −0.01 | | | Yes | Yes | 0.101 | All |
| *3.97* | *−0.93* | *3.39* | *−3.13* | *1.32* | *−1.67* | *−1.89* | | | | | | |
| −0.01 | 0.03 | 0.00 | 0.01 | 0.01 | −0.59 | −0.02 | | | Yes | Yes | 0.102 | HF |
| *−0.77* | *1.51* | *−0.52* | *0.29* | *0.66* | *−1.99* | *−1.17* | | | | | | |
| −0.01 | 0.03 | 0.00 | 0.00 | 0.01 | −0.59 | −0.02 | 0.00 | | Yes | Yes | 0.105 | HF |
| *−0.70* | *1.49* | *−0.40* | *0.19* | *0.56* | *−1.97* | *−1.12* | *−0.68* | | | | | |
| 0.01 | 0.02 | 0.01 | −0.02 | 0.00 | −0.23 | 0.00 | | | Yes | Yes | 0.227 | OI |
| *1.45* | *1.72* | *1.74* | *−1.85* | *0.19* | *−1.66* | *−0.18* | | | | | | |
| 0.02 | 0.02 | 0.01 | −0.01 | 0.00 | −0.21 | 0.00 | | 0.00 | Yes | Yes | 0.233 | OI |
| *2.09* | *1.80* | *1.29* | *−1.72* | *0.12* | *−1.64* | *−0.02* | | *1.08* | | | | |

**Table 2**
Data summary statistics.

| Panel A: Cross-section | Mean | SD | 1st percentile | Median | 99th percentile |
|---|---|---|---|---|---|
| **Realized betas** | | | | | |
| $\beta^{MKT}$ | 1.083 | 0.551 | 0.066 | 1.008 | 2.634 |
| $\beta^{DR}$ | 0.022 | 0.396 | −0.964 | 0.011 | 1.099 |
| $\beta^{VIX}$ | 0.032 | 0.334 | −0.810 | 0.023 | 0.942 |
| **Predicted betas** | | | | | |
| $\beta^{MKT}$ | 1.024 | 0.331 | 0.344 | 0.998 | 1.821 |
| $\beta^{DR}$ | 0.035 | 0.077 | −0.124 | 0.030 | 0.230 |
| $\beta^{VIX}$ | 0.023 | 0.031 | −0.049 | 0.023 | 0.096 |
| **Controls** | | | | | |
| Idiovol | 0.024 | 0.010 | 0.009 | 0.023 | 0.052 |
| CSK | −0.181 | 0.164 | −0.566 | −0.178 | 0.191 |
| log(Size) | 6.169 | 1.739 | 2.721 | 6.010 | 10.766 |
| BTM | 0.580 | 0.460 | 0.045 | 0.487 | 2.127 |

| Panel B: Time series | Mean | SD | Average autocorrelation |
|---|---|---|---|
| **Realized betas** | | | |
| $\beta^{MKT}$ | 1.120 | 0.399 | 0.833 |
| $\beta^{DR}$ | 0.067 | 0.381 | 0.741 |
| $\beta^{VIX}$ | 0.031 | 0.327 | 0.768 |

## 4. Beta predictability and the price of risk

- This section serves to quantify reductions in risk premiums induced by the practical challenges faced by investors when predicting betas.

- We apply a standard multi factor model specification:

$$E_t\left(r_{t+1}^{j}\right)=\beta_{i,t}^{M}\lambda_{m,t}+\sum_{k=1}^{K}\beta_{i,t}^{k}\lambda_{k,t}$$

$\beta_{i,t}^{M}$ : stock's market beta ; $\lambda_{m,t}$ : the price of market risk
$\beta_{i,t}^{k}$ : additional risk factor exposures ; $\lambda_{k,t}$: corresponding prices of risk.
Betas are conditional on information observed by the investor at t .

- Substantial look-ahead biases may arise lagged or predicted betas often differ substantially from ex-post realized ones. Portfolio sorts or asset pricing tests using the three different types of betas may lead to very different outcomes.
- We predict that prices of risk obtained using lagged or predicted betas are lower than risk premium estimates using ex-post realized betas.

## 4.1 Portfolio sorts

**Table 3**

Beta-sorted portfolios.

Panel A: Relative downside beta sorts

| Ex-post realized beta | | | | Predicted beta | | | | Lagged beta | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Portfolio | Raw return | CAPM alpha | $\beta^{DR}$ | Portfolio | Raw return | CAPM alpha | $\beta^{DR}$ | Portfolio | Raw return | CAPM alpha | $\beta^{DR}$ |
| Lo $\beta$ | 10.34% | 1.17% | −0.51 | Lo $\beta$ | 9.48% | 2.14% | −0.03 | Lo $\beta$ | 13.24% | 5.58% | −0.02 |
| 2 | 12.38% | 5.38% | −0.16 | 2 | 10.99% | 4.60% | −0.01 | 2 | 11.38% | 4.93% | −0.01 |
| 3 | 13.50% | 6.97% | 0.02 | 3 | 11.94% | 5.52% | 0.01 | 3 | 11.00% | 4.78% | 0.01 |
| 4 | 16.20% | 9.34% | 0.20 | 4 | 13.66% | 7.24% | 0.02 | 4 | 11.44% | 4.81% | 0.03 |
| Hi $\beta$ | 23.42% | 14.53% | 0.59 | Hi $\beta$ | 18.51% | 11.90% | 0.04 | Hi $\beta$ | 12.45% | 4.35% | 0.05 |
| Hi-lo | 13.08% | 13.36% | 1.09 | Hi-lo | 9.03% | 9.76% | 0.08 | Hi-lo | −0.79% | −1.23% | 0.07 |
| *t*-stat | *5.71* | *5.75* | | *t*-stat | *4.30* | *5.11* | *4.86* | *t*-stat | *−0.73* | *−1.00* | *8.65* |

Panel B: VIX beta sorts

| Ex-post realized beta | | | | Predicted beta | | | | Lagged beta | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Portfolio | Raw return | CAPM alpha | $\beta^{VIX}$ | Portfolio | Raw return | CAPM alpha | $\beta^{VIX}$ | Portfolio | Raw return | CAPM alpha | $\beta^{VIX}$ |
| Lo $\beta$ | 20.46% | 12.66% | −0.41 | Lo $\beta$ | 16.87% | 9.57% | 0.01 | Lo $\beta$ | 12.88% | 6.15% | 0.01 |
| 2 | 15.10% | 8.73% | −0.12 | 2 | 14.07% | 7.21% | 0.02 | 2 | 11.52% | 5.32% | 0.01 |
| 3 | 13.62% | 7.06% | 0.03 | 3 | 11.33% | 5.02% | 0.03 | 3 | 11.47% | 5.17% | 0.02 |
| 4 | 13.18% | 5.66% | 0.18 | 4 | 11.33% | 5.01% | 0.03 | 4 | 11.68% | 4.78% | 0.04 |
| Hi $\beta$ | 13.48% | 3.26% | 0.51 | Hi $\beta$ | 10.98% | 4.60% | 0.05 | Hi $\beta$ | 11.96% | 3.03% | 0.07 |
| Hi-lo | −6.98% | −9.40% | 0.92 | Hi-lo | −5.89% | −4.97% | 0.03 | Hi-lo | −0.92% | −3.12% | 0.06 |
| *t*-stat | *−2.14* | *−2.94* | | *t*-stat | *−1.46* | *−1.57* | *2.04* | *t*-stat | *−0.64* | *−1.89* | *3.76* |

**Table 4**
Hedge portfolio beta characteristics.

| Panel A: Lagged beta hedge portfolio | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Conf Int | 99% | | 95% | | 90% | | 80% | |
| | Mean | LB | UB | LB | UB | LB | UB | LB | UB |
| $\beta^{DR}$ | 0.07 | −0.11 | 0.23 | −0.07 | 0.19 | −0.03 | 0.18 | 0.00 | 0.16 |
| $\beta^{VIX}$ | 0.04 | −0.15 | 0.44 | −0.08 | 0.36 | −0.05 | 0.26 | −0.03 | 0.11 |

| Panel B: Predicted beta hedge portfolio | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | CI | 99% | | 95% | | 90% | | 80% | |
| | Mean | LB | UB | LB | UB | LB | UB | LB | UB |
| $\beta^{DR}$ | 0.08 | −0.14 | 0.26 | −0.12 | 0.23 | −0.09 | 0.21 | −0.06 | 0.19 |
| $\beta^{VIX}$ | 0.03 | −0.37 | 0.32 | −0.30 | 0.25 | −0.16 | 0.18 | −0.07 | 0.14 |

| | Lo IV | 2 | 3 | 4 | Hi IV | Hi - lo | t-stat |
|---|---|---|---|---|---|---|---|
| **Panel A: Relative downside beta** | | | | | | | |
| **Ex-post realized** | | | | | | | |
| Lo β | 2.59% | −0.25% | −1.02% | −2.99% | 0.21% | −2.38% | *−0.48* |
| 2 | 4.55% | 3.27% | 4.68% | 3.91% | 3.45% | −1.10% | *−0.27* |
| 3 | 5.23% | 4.64% | 5.35% | 6.86% | 6.13% | 0.90% | *0.25* |
| 4 | 5.89% | 6.02% | 6.76% | 9.08% | 9.24% | 3.35% | *0.89* |
| Hi β | 6.09% | 6.57% | 9.50% | 10.63% | 9.97% | 3.88% | *0.87* |
| Hi - lo | 3.49% | 6.82% | 10.53% | 13.63% | 9.76% | 6.26% | *2.31* |
| t-stat | *2.20* | *3.74* | *4.30* | *4.17* | *4.63* | *2.31* | |
| **Predicted beta** | | | | | | | |
| Lo β | 3.42% | 1.40% | 0.35% | 1.29% | 0.93% | −2.49% | *−0.51* |
| 2 | 5.10% | 3.34% | 3.45% | 2.93% | 4.39% | −0.70% | *−0.16* |
| 3 | 4.77% | 4.49% | 4.56% | 5.36% | 3.75% | −1.02% | *−0.29* |
| 4 | 5.43% | 4.61% | 7.02% | 7.54% | 8.36% | 2.93% | *0.80* |
| Hi β | 6.08% | 6.26% | 8.42% | 8.28% | 10.50% | 4.42% | *1.24* |
| Hi - lo | 2.66% | 4.86% | 8.07% | 6.99% | 9.57% | 6.90% | *2.38* |
| t-stat | *1.98* | *2.44* | *2.43* | *2.02* | *2.92* | *2.38* | |
| **Lagged beta** | | | | | | | |
| Lo β | 5.41% | 4.54% | 5.04% | 5.49% | 7.28% | 1.87% | *0.44* |
| 2 | 5.19% | 4.33% | 4.58% | 4.65% | 6.25% | 1.06% | *0.27* |
| 3 | 5.23% | 3.72% | 4.27% | 5.35% | 5.59% | 0.35% | *0.09* |
| 4 | 4.68% | 3.51% | 5.23% | 4.72% | 4.88% | 0.21% | *0.06* |
| Hi β | 4.30% | 4.00% | 4.67% | 5.20% | 3.93% | −0.37% | *−0.10* |
| Hi - lo | −1.10% | −0.54% | −0.36% | −0.28% | −3.35% | −2.24% | *−1.22* |
| t-stat | *−1.98* | *−0.53* | *−0.27* | *−0.26* | *−1.89* | *−1.22* | |

Table 5
Double-sorted portfolios
CAPM alphas.

Table 5
Double-sorted portfolios
CAPM alphas.

Panel B: VIX beta

Ex-post realized beta

| | Lo IV | 2 | 3 | 4 | Hi IV | Hi - lo | *t*-stat |
|---|---|---|---|---|---|---|---|
| Lo $\beta$ | 7.36% | 7.94% | 9.85% | 11.38% | 10.21% | 2.85% | *0.82* |
| 2 | 6.59% | 5.80% | 7.12% | 8.52% | 9.14% | 2.55% | *0.69* |
| 3 | 5.39% | 4.85% | 5.35% | 6.50% | 5.87% | 0.48% | *0.13* |
| 4 | 3.98% | 2.59% | 3.52% | 3.32% | 3.27% | −0.72% | *−0.17* |
| Hi $\beta$ | 1.03% | −0.91% | −0.56% | −2.22% | 0.51% | −0.51% | *−0.09* |
| Hi - lo | −6.33% | −8.85% | −10.40% | −13.61% | −9.69% | −3.36% | *−0.97* |
| *t*-stat | *−4.88* | *−5.74* | *−3.80* | *−4.70* | *−2.84* | *−0.97* | |

Predicted beta

| | Lo IV | 2 | 3 | 4 | Hi IV | Hi - lo | *t*-stat |
|---|---|---|---|---|---|---|---|
| Lo $\beta$ | 4.93% | 5.56% | 7.40% | 8.93% | 8.98% | 4.05% | *0.99* |
| 2 | 4.92% | 4.74% | 6.93% | 6.95% | 5.93% | 1.00% | *0.25* |
| 3 | 5.73% | 4.42% | 5.24% | 5.23% | 4.39% | −1.34% | *−0.33* |
| 4 | 5.34% | 3.42% | 3.15% | 3.48% | 3.79% | −1.55% | *−0.34* |
| Hi $\beta$ | 3.88% | 1.95% | 1.07% | 0.81% | 4.86% | 0.98% | *0.28* |
| Hi - lo | −1.05% | −3.61% | −6.32% | −8.12% | −4.12% | −3.07% | *−1.13* |
| *t*-stat | *−0.83* | *−2.57* | *−3.39* | *−3.23* | *−1.66* | *−1.13* | |

Lagged beta

| | Lo IV | 2 | 3 | 4 | Hi IV | Hi - lo | *t*-stat |
|---|---|---|---|---|---|---|---|
| Lo $\beta$ | 4.78% | 4.09% | 5.45% | 5.55% | 6.47% | 1.69% | *0.49* |
| 2 | 4.93% | 3.68% | 5.11% | 6.11% | 5.45% | 0.52% | *0.14* |
| 3 | 4.77% | 4.54% | 5.56% | 5.87% | 6.64% | 1.87% | *0.47* |
| 4 | 4.84% | 3.87% | 4.64% | 4.53% | 5.04% | 0.21% | *0.05* |
| Hi $\beta$ | 5.49% | 3.91% | 3.02% | 3.34% | 4.34% | −1.15% | *−0.25* |
| Hi - lo | 0.72% | −0.18% | −2.43% | −2.22% | −2.13% | −2.84% | *−1.05* |
| *t*-stat | *0.98* | *−0.19* | *−1.95* | *−1.92* | *−0.89* | *−1.05* | |

## 4.2 Cross-sectional regressions

- This section studies the risk premiums associated with ex-ante versus realized betas through a Fama-MacBeth analysis.
- Prices of risk are estimated using the regression specification below and results are reported in Table 6

$$r_{i,t} = \lambda_{m,t}\beta_{i,t}^{M} + \sum_{k=1}^{K}\lambda_{k,t}\beta_{i,t}^{k} + \gamma_t X_{i,t-1} + \varepsilon_{i,t}$$

$\beta_{i,t}^{M}$ and $\beta_{i,t}^{k}$ represent market, relative downside , and VIX risk exposure.

- We test the hypothesis that risk premium estimates are zero (or small) for betas that are hard to predict. Hence, we expect low prices of risk for predicted and lagged downside and VIX betas.

- Hedging demand decreases when betas are more difficult to predict.

- Risk premium estimates are small and mostly insignificant when we take into account that an investor cannot observe, but needs to, predict future risk exposure.

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# Table 6 Cross-sectional regressions.

Panel B: Predicted betas

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\beta^{MKT}$ | | 0.002 | −0.017 | 0.002 | 0.006 | 0.005 | | −0.001 | −0.015 | −0.001 | 0.008 | 0.003 |
| *t*-stat | | *0.13* | *−1.31* | *0.11* | *0.37* | *0.31* | | *−0.09* | *−0.83* | *−0.07* | *0.54* | *0.26* |
| $\beta^{DR}$ | 0.020 | 0.016 | −0.002 | 0.016 | 0.001 | 0.014 | | | | | | |
| *t*-stat | *2.69* | *2.27* | *−0.26* | *2.11* | *0.10* | *2.02* | | | | | | |
| $\beta^{VIX}$ | | | | | | | −0.028 | −0.017 | −0.015 | −0.017 | −0.029 | −0.020 |
| *t*-stat | | | | | | | *−2.01* | *−1.64* | *−1.45* | *−1.70* | *−3.05* | *−1.88* |
| IV | | | 0.030 | | | | | | 0.015 | | | |
| *t*-stat | | | *2.04* | | | | | | *1.80* | | | |
| CSK | | | | 0.005 | | | | | | 0.003 | | |
| *t*-stat | | | | *1.54* | | | | | | *0.86* | | |
| log(Size) | | | | | −0.029 | | | | | | −0.020 | |
| *t*-stat | | | | | *−2.64* | | | | | | *−2.57* | |
| BTM | | | | | | 0.015 | | | | | | 0.017 |
| *t*-stat | | | | | | *2.22* | | | | | | *2.35* |
| $R^2$ | 0.010 | 0.043 | 0.047 | 0.045 | 0.049 | 0.050 | 0.021 | 0.042 | 0.048 | 0.043 | 0.05 | 0.05 |

Panel A: Ex-post realized betas

| | (1) | (2) | (3) | (4) | (5) | (6) |
|---|---|---|---|---|---|---|
| $\beta^{MKT}$ | | 0.043 | 0.057 | | 0.044 | 0.057 |
| *t*-stat | | *1.89* | *2.46* | | *1.97* | *2.53* |
| $\beta^{DR}$ | 0.038 | 0.030 | 0.028 | | | |
| *t*-stat | *4.54* | *4.91* | *4.84* | | | |
| $\beta^{VIX}$ | | | | −0.026 | −0.026 | −0.029 |
| *t*-stat | | | | *−2.40* | *−4.04* | *−5.34* |
| IV | | | −0.016 | | | −0.013 |
| *t*-stat | | | *−2.29* | | | *−1.79* |
| CSK | | | 0.004 | | | 0.004 |
| *t*-stat | | | *1.35* | | | *1.27* |
| log(Size) | | | −0.031 | | | −0.032 |
| *t*-stat | | | *−4.1* | | | *−4.07* |
| BTM | | | 0.022 | | | 0.021 |
| *t*-stat | | | *3.72* | | | *3.59* |
| $R^2$ | 0.017 | 0.074 | 0.100 | 0.022 | 0.074 | 0.101 |

Panel C: Lagged betas

| | (1) | (2) | (3) | (4) | (5) | (6) |
|---|---|---|---|---|---|---|
| $\beta^{MKT}$ | | 0.000 | −0.008 | | 0.000 | −0.006 |
| *t*-stat | | *0.00* | *−0.94* | | *−0.01* | *−0.73* |
| $\beta^{DR}$ | −0.003 | −0.004 | −0.007 | | | |
| *t*-stat | *−0.76* | *− 1.05* | *−1.61* | | | |
| $\beta^{VIX}$ | | | | −0.005 | 0.001 | −0.003 |
| *t*-stat | | | | *−0.78* | *0.21* | *−1.10* |
| IV | | | 0.014 | | | 0.014 |
| *t*-stat | | | *1.12* | | | *1.14* |
| CSK | | | −0.003 | | | 0.001 |
| *t*-stat | | | *−1.03* | | | *0.41* |
| log(Size) | | | −0.012 | | | −0.012 |
| *t*-stat | | | *−1.70* | | | *−1.74* |
| BTM | | | 0.010 | | | 0.010 |
| *t*-stat | | | *1.34* | | | *1.38* |
| $R^2$ | 0.004 | 0.031 | 0.056 | 0.006 | 0.03 | 0.056 |

# 5. Other risk factors

Table 7 Beta predictability and risk premiums for other risk factors

(2.17-0.7)/2.17=67.65%

Result:
Investors who want to hedge against Fama-French risk or changes in the investment opportunity set face similar difficulties as investors who want to hedge against crashes or changes in volatility.

SMB beta: ex-post realized

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{SMB}$ |
|---|---|---|---|
| Hi-lo | 2.99% | 0.61% | 2.17 |
| *t-stat* | *0.89* | *0.19* | |
| $\Delta\beta_{spread}$ | −67.65% | | |

SMB beta: lagged

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{HML}$ |
|---|---|---|---|
| Hi-lo | 3.59% | 2.19% | 0.70 |
| *t-stat* | *2.41* | *1.51* | *3.5* |

HML beta: ex-post realized

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{HML}$ |
|---|---|---|---|
| Hi-lo | −8.22% | −5.81% | 2.69 |
| *t-stat* | *−1.92* | *−1.43* | |
| $\Delta\beta_{spread}$ | −78.38% | | |

HML beta: lagged

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{HML}$ |
|---|---|---|---|
| Hi-lo | 0.52% | 1.88% | 0.58 |
| *t-stat* | *0.22* | *0.85* | *−3.6* |

RMW beta: ex-post realized

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{RMW}$ |
|---|---|---|---|
| Hi-lo | −3.60% | 0.56% | 3.08 |
| *t-stat* | *−0.93* | *0.15* | |
| $\Delta\beta_{spread}$ | −84.11% | | |

RMW beta: lagged

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{RMW}$ |
|---|---|---|---|
| Hi-lo | 2.53% | 3.95% | 0.49 |
| *t-stat* | *1.42* | *1.55* | *−3.43* |

CMA beta: ex-post realized

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{CMA}$ |
|---|---|---|---|
| Hi-lo | −1.32% | 1.30% | 3.42 |
| *t-stat* | *0.59* | *0.55* | |
| $\Delta\beta_{spread}$ | −82.93% | | |

CMA beta: lagged

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{CMA}$ |
|---|---|---|---|
| Hi-lo | −0.60% | 1.07% | 0.58 |
| *t-stat* | *−0.41* | *0.78* | *−8.27* |

Term spread beta: ex-post realized

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{TS}$ |
|---|---|---|---|
| Hi-lo | −0.98% | −4.61% | 1.25 |
| *t-stat* | *−0.35* | *−1.79* | |
| $\Delta\beta_{spread}$ | −89.71% | | |

Term spread beta: lagged

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{TS}$ |
|---|---|---|---|
| Hi-lo | 0.97% | −0.25% | 0.13 |
| *t-stat* | *0.65* | *−0.17* | *5.26* |

Credit spread beta: ex-post realized

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{CS}$ |
|---|---|---|---|
| Hi-lo | −0.50% | 1.05% | 2.56 |
| *t-stat* | *−0.24* | *0.56* | |
| $\Delta\beta_{spread}$ | −95.11% | | |

Credit spread beta: lagged

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{CS}$ |
|---|---|---|---|
| Hi-lo | −1.04% | 0.29% | 0.13 |
| *t-stat* | *−0.91* | *0.25* | *−2.51* |

Δ Oil price beta: ex-post realized

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{oil}$ |
|---|---|---|---|
| Hi-lo | 2.90% | 0.68% | 1.23 |
| *t-stat* | *0.73* | *0.18* | |
| $\Delta\beta_{spread}$ | −82.59% | | |

Δ Oil price beta: lagged

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{oil}$ |
|---|---|---|---|
| Hi-lo | 0.40% | −1.08% | 0.21 |
| *t-stat* | *0.24* | *−0.53* | *2.44* |

Rf beta: ex-post realized

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{Rf}$ |
|---|---|---|---|
| Hi-lo | 1.96% | 3.06% | 0.26 |
| *t-stat* | *1.1* | *1.94* | |
| $\Delta\beta_{spread}$ | −99.65% | | |

Rf beta: lagged

| Portfolio | *Ret* | $\alpha^{CAPM}$ | $\beta^{Rf}$ |
|---|---|---|---|
| Hi-lo | −1.58% | −1.81% | 0.00 |
| *t-stat* | *−1.6* | *−1.88* | *0.4* |

## 6. Why are ex-post realized betas priced?

- Estimation error in betas induces an upward biases in risk premiums estimates.
- Under the null of the CAPM we have

$$r_{it} = \beta_{it}\lambda_t + \epsilon_{it}. \tag{23}$$

Betas are not observable but have to be estimated:

$$r_{it} = \hat{\beta}_{it}\lambda_t + \eta_{it}, \tag{24}$$

with

$$\hat{\beta}_{it} = \beta_{it} + B_i,$$

$B_i$ : the estimation error for the beta.

The error teams:

$$\eta_{it} = \epsilon_{it} - B_i\lambda_t. \tag{25}$$

Risk premium estimate

$$\hat{\lambda}_t = \frac{cov^c(R_{it}, \hat{\beta}_{it})}{var(\hat{\beta}_{it})} = \lambda_t + \frac{cov^c(\eta_{it}, \hat{\beta}_{it})}{var(\hat{\beta}_{it})}. \tag{27}$$

$\mathrm{cov}^c$ : cross-sectional covariance.

the missing period : the average of betas estimated over period t − 1 and period t + 1

**Table 8**
Fama-MacBeth missing period betas.

| | (1) | (2) | (3) | (4) | (5) | (6) |
|---|---|---|---|---|---|---|
| $\beta^{MKT}$ | | 0.062 | 0.078 | | 0.062 | 0.079 |
| *t*-stat | | *2.52* | *3.00* | | *2.89* | *3.43* |
| $\beta^{DR}$ | 0.002 | 0.007 | 0.008 | | | |
| *t*-stat | *0.32* | *1.16* | *1.27* | | | |
| $\beta^{VIX}$ | | | | 0.006 | −0.003 | −0.007 |
| *t*-stat | | | | *0.35* | *−0.28* | *−0.87* |
| IV | | | −0.022 | | | −0.020 |
| *t*-stat | | | *−2.86* | | | *−2.56* |
| CSK | | | 0.008 | | | 0.005 |
| *t*-stat | | | *1.91* | | | *1.47* |
| log(Size) | | | −0.046 | | | −0.044 |
| *t*-stat | | | *−5.67* | | | *−5.62* |
| BTM | | | 0.023 | | | 0.024 |
| *t*-stat | | | *3.97* | | | *4.20* |
| $R^2$ | 0.009 | 0.072 | 0.103 | 0.02 | 0.071 | 0.102 |

The smaller missing period risk premium estimates are in line with the estimates produced by predicted or lagged betas.

## 6.1 Simulations

- To add further credibility to the bias explanation, we also run a simulation exercise, where we simulate 100 panels of returns under the null of the CAPM

$$r_{it}^{(s)} = \beta_{it} R_{Mt} + \epsilon_{it}, \quad (28)$$

using realized betas estimated over one year of daily data from our sample as the true betas $\beta_{it}$.
Error terms($\epsilon_{it}$) independently from a normal distribution with mean zero and standard deviation $\sigma_{\epsilon}$ = 0.5 .
Cross-sectional correlation between estimated betas$\hat{\beta}_{it} = \beta_{it} + B_{it}$ and time series error terms ($\epsilon_{it}$) is created as follows

$$B_{it} = \rho \epsilon_{it} + (1 - \rho) v_i, \quad (29)$$

$v_i$ is drawn from a normal distribution with mean zero and standard deviation $\sigma_v$ = 0.5 .

average risk premium estimates

time series averages of intercepts

| | $\hat{\bar{\lambda}}_0$ | $\hat{\bar{\lambda}}_1$ | $R^2$ |
|---|---|---|---|
| $B_i = 0$ | | | |
| | 0.0000 | 0.0772 | 3.81% |
| | *0.05* | *3.12* | |
| $B_i \neq 0, \quad \rho = 0$ | | | |
| | 0.0380 | 0.0422 | 2.22% |
| | *3.03* | *3.13* | |
| $B_i \neq 0, \quad \rho = 0.05$ | | | |
| | −0.0164 | 0.0928 | 3.60% |
| | *−1.37* | *6.61* | |
| $B_i \neq 0, \quad \rho = 0.2$ | | | |
| | −0.1891 | 0.2531 | 13.88% |
| | *−13.24* | *15.05* | |

**Table 9**
Simulating biases in risk premium estimates.

This table reports average risk premium estimates $\hat{\bar{\lambda}}_1$ using Fama-MacBeth analysis across 100 panels of returns. Where returns are simulated under the null of the CAPM:

$$R_{it} = \beta_{it}\mu + \epsilon_{it},$$

where $\mu$ equals the sample mean of 0.0772 and $\epsilon_{it}$ are independent draws from a normal distribution with mean zero and standard deviation 0.5. We use realized betas estimated over one year of daily data as proxies for the true beta ($\beta_{it}$). Cross-sectional correlation between estimated betas $\hat{\beta}_{it} = \beta_{it} + B_i$ and time series error terms ($\epsilon_{it}$) are created as follows

$$B_i = \rho\tilde{\epsilon} + (1-\rho)\nu_i,$$

where $\nu_i$ are independent draws from a normal distribution with mean zero and standard deviation 0.5 and $\tilde{\epsilon}$ represents the cross-sectionally standardized error terms from the time series regression $\epsilon_{it}$. We report time series averages of intercepts $\hat{\bar{\lambda}}_0$ and risk premium estimates $\hat{\bar{\lambda}}_1$ with corresponding $t$-statistics (in italics) and $R^2$s for various values of $\rho$.

These findings corroborate the importance of incorporating period t in the estimation of betas and illustrate that risk premium estimates can be biased even under mild conditions.

## 7. Hedge portfolios from the investor's perspective

Our theoretical setting conjectures that higher beta uncertainty should lead to lower partial equilibrium demands for hedge assets.

A representative investor maximizes expected utility of terminal wealth by allocating between the market portfolio and a long-short (hedge) portfolio

$$\max_{\alpha_E,\alpha_H} E[U(W_T)], \tag{30}$$

where terminal wealth is given by

$$W_T = W_0\left[r_f + \alpha_E\left(r_E - r_f\right) + \alpha_H r_H\right]. \tag{31}$$

E and H refer to returns and weights for the equity market portfolio and the hedge portfolio, respectively.

We use the Generalized Method of Moments (GMM) to estimate the optimal unconditional weights.

$$\begin{cases} E\left[U'(W_T)\left(r_E - r_f\right)\right] = 0 \\ E\left[U'(W_T)(r_H)\right] = 0. \end{cases} \tag{32}$$

We derive the following moment conditions:

$$\begin{cases} E\left[W_T^{-\gamma}\left(r_E - r_f\right)\right] = 0 \\ E\left[W_T^{-\gamma}(r_H)\right] = 0, \end{cases} \tag{33}$$

and use GMM to estimate weights satisfying these moment conditions.

**Table 10**
Portfolio optimization for CRRA investors.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Panel A: Market only | | | | | | | | | |
| $\gamma$ | 1 | 2 | 5 | 10 | 20 | | | | | |
| $w_M$ | 1.92 | 1.16 | 0.5 | 0.25 | 0.13 | | | | | |
| $t$-stat | *4.25* | *2.67* | *2.32* | *2.22* | *2.14* | | | | | |
| | Panel B: Ex-post realized betas | | | | | | | | | |
| | $\beta^{DR}$ hedge portfolio | | | | | $\beta^{VIX}$ hedge portfolio | | | | |
| $\gamma$ | 1 | 2 | 5 | 10 | 20 | 1 | 2 | 5 | 10 | 20 |
| $w_M$ | 1.73 | 1.01 | 0.46 | 0.23 | 0.11 | 2.15 | 1.36 | 0.59 | 0.3 | 0.15 |
| $t$-stat | *19.69* | *2.69* | *1.89* | *1.76* | *1.69* | *5.56* | *3.05* | *2.57* | *2.44* | *2.35* |
| $w_H$ | 8.06 | 5.96 | 2.61 | 1.26 | 0.59 | −0.8 | −0.55 | −0.26 | −0.14 | −0.08 |
| $t$-stat | *50.33* | *8.98* | *5.88* | *5.42* | *4.99* | *−6.49* | *−2.93* | *−2.32* | *−2.29* | *−2.5* |
| | Panel C: Predicted betas | | | | | | | | | |
| | $\beta^{DR}$ hedge portfolio | | | | | $\beta^{VIX}$ hedge portfolio | | | | |
| $\gamma$ | 1 | 2 | 5 | 10 | 20 | 1 | 2 | 5 | 10 | 20 |
| $w_M$ | 1.74 | 1.83 | 1.02 | 0.56 | 0.32 | 1.71 | 1.06 | 0.46 | 0.24 | 0.12 |
| $t$-stat | *5.14* | *4.49* | *3.93* | *4.09* | *4.54* | *3.54* | *2.51* | *2.24* | *2.17* | *2.13* |
| $w_H$ | 6.84 | 5.08 | 2.53 | 1.37 | 0.77 | −1.25 | −0.66 | −0.25 | −0.11 | −0.05 |
| $t$-stat | *9.38* | *7.97* | *6.57* | *6.30* | *6.48* | *−1.85* | *−1.57* | *−1.47* | *−1.33* | *−1.04* |
| | Panel D: Lagged betas | | | | | | | | | |
| | $\beta^{DR}$ hedge portfolio | | | | | $\beta^{VIX}$ hedge portfolio | | | | |
| $\gamma$ | 1 | 2 | 5 | 10 | 20 | 1 | 2 | 5 | 10 | 20 |
| $w_M$ | 2.07 | 1.25 | 0.53 | 0.27 | 0.13 | 1.94 | 1.26 | 0.56 | 0.28 | 0.14 |
| $t$-stat | *4.41* | *2.79* | *2.41* | *2.31* | *2.22* | *5.23* | *2.9* | *2.36* | *2.20* | *2.08* |
| $w_H$ | −1.63 | −0.92 | −0.39 | −0.21 | −0.12 | −1.62 | −1.06 | −0.46 | −0.23 | −0.1 |
| $t$-stat | *−1.40* | *−1.19* | *−1.15* | *−1.18* | *−1.27* | *−1.12* | *−1.28* | *−1.27* | *−1.20* | *−1.05* |

We cannot rule out that the relative downside beta premiums constitute compensations for correlated risk factors or characteristics.

## 8. Conclusion

- Investors are less willing to use an asset for hedging purposes if it is uncertain how well the asset hedges in the future . Risk exposure is indeed difficult to predict for downside risk and volatility risk.

- Lagged betas are only mildly (and some times negatively) correlated with future betas. Moreover, downside and volatility risk seems closely related to idiosyncratic volatility and firm size.

- Portfolios sorted on lagged or predicted betas earn significantly lower premiums and have ex-post realized exposures close to zero.

- Risk premiums also decline substantially after controlling for idiosyncratic volatility and firm size.

- We run a portfolio optimization for a CRRA agent who allocates wealth between the market and a hedge portfolio in zero net supply. Representative agents allocate less wealth to portfolios formed using investable (lagged or predicted) betas vis-vis non-investable ex-post realized betas.

谢谢大家！